MATROIDS AND GRAPHS

BY
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1. **Introduction.** This paper is a sequel to two others, [3] and [4], recently published in these Transactions. It also uses the definitions and theorems of [2]. We refer to these three papers as HI, HII and A respectively.

In the original paper on matroids, [6], Hassler Whitney pointed out that the circuits of any finite graph G define a matroid. We call this the *circuit-matroid* and its dual the *bond-matroid* of G. In the present paper we determine a necessary and sufficient condition, in terms of matroid structure, for a given matroid $\boldsymbol{M}$ to be *graphic* (*cographic*), that is the bond-matroid (circuit-matroid) of some finite graph. The condition is that $\boldsymbol{M}$ shall be regular and shall not contain, in a sense to be explained, the circuit-matroid (bond-matroid) of a Kuratowski graph, that is a graph with one of the structures shown in Figure I.

Some of the intermediate results seem to be of interest in themselves. These include the theory of dual matroids in §2, Theorem (7.3) on regular matroids, and Theorem (8.4) on the bond-matroids of graphs.

Our main theorem is evidently closely related to the theorem of Kuratowski on planar graphs [1]. This states that a graph is planar (i.e., can be imbedded in the plane) if and only if it contains no graph with the point set structure of a Kuratowski graph. Indeed it is not difficult to prove Kuratowski's Theorem from ours, using the principle that a graph is planar if and only if it has a dual graph, that is if and only if its circuit-matroid is graphic. However this paper is long enough already and we refrain from adding matter not essential to the proof and understanding of the main theorem.

2. **Dual matroids.** We define a matroid $\boldsymbol{M}$ on a set M, its flats and their dimensions as in HI, §1. We call the dimension of the largest flat $\langle M\rangle$ also the dimension $d\boldsymbol{M}$ of the matroid. (See HI, §2, for the notation $\langle S\rangle$). We write $\alpha(S)$ for the number of elements of any finite set S. We proceed to give a definition of the dual of $\boldsymbol{M}$ analogous to the definition of a dual vector space in terms of orthogonality.

First, in analogy with A, §4, we define a *dendroid* of $\boldsymbol{M}$ as a minimal subset D of M which meets every $X\in\boldsymbol{M}$. Then for each $a\in D$ there is a point $J(\boldsymbol{M}, D, a)$ of $\boldsymbol{M}$ such that $D\cap J(\boldsymbol{M}, D, a)=\{a\}$. Moreover $J(\boldsymbol{M}, D, a)$ is unique, for if $(X, Y)\in\boldsymbol{M}$, $X\cap D=Y\cap D=\{a\}$ and $X\neq Y$ it follows by Axiom II of the definition of a matroid that there exists $Z\in\boldsymbol{M}$ such that $Z\subseteq(X\cup Y)-\{a\}\subseteq M-D$.

(2.1) *If $a\in X\in\boldsymbol{M}$ there exists a dendroid D of $\boldsymbol{M}$ such that $X=J(\boldsymbol{M}, D, a)$.*

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Proof. By Axiom I $(M-X)\cup\{a\}$ meets every $Y\in\boldsymbol{M}$. It thus has a dendroid D of $\boldsymbol{M}$ as a subset. But $a\in D$ since $D\cap X\neq\emptyset$. Hence $X=J(\boldsymbol{M},D,a)$.

(2.2) *If D is a dendroid of $\boldsymbol{M}$ then $\alpha(D)=d\boldsymbol{M}+1$.*

Proof. If $D=\emptyset$, then $\boldsymbol{M}$ is a null class and $d\boldsymbol{M}=-1$. In the remaining case we enumerate the cells of D as $a_1,\cdots,a_k$ and write $S_0=\langle M\rangle$, $S_i=\langle M-\{a_1,\cdots,a_i\}\rangle$ $(i=1,\cdots,k)$. It is clear that $J(\boldsymbol{M},D,a_i)\subseteq S_{i-1}$ and so $a_i\in S_{i-1}$ and $S_i=\langle S_{i-1}-\{a_i\}\rangle$. But $dS_k=-1$ since $D\cap S_k=\emptyset$. Hence $d\boldsymbol{M}=dS_0=\alpha(D)-1$, by HI, (2.3).

(2.3) *If $\boldsymbol{M}'$ is a matroid on M having the same dendroids as $\boldsymbol{M}$, then $\boldsymbol{M}'=\boldsymbol{M}$.*

Proof. Let D be any dendroid of $\boldsymbol{M}$ and $\boldsymbol{M}'$, and a any cell of D. Suppose $a\neq b\in J(\boldsymbol{M},D,a)$. Then $(D-\{a\})\cup\{b\}$ meets every $X\in\boldsymbol{M}$ and so contains a dendroid E of $\boldsymbol{M}$. Clearly $b\in E$. Since $J(\boldsymbol{M}',D,a)$ meets E we have $b\in J(\boldsymbol{M}',D,a)$. We deduce that $J(\boldsymbol{M},D,a)\subseteq J(\boldsymbol{M}',D,a)$. Similarly $J(\boldsymbol{M}',D,a)\subseteq J(\boldsymbol{M},D,a)$ and therefore $J(\boldsymbol{M},D,a)=J(\boldsymbol{M}',D,a)$. The theorem follows by (2.1).

We call two subsets S and T of M *orthogonal* if $\alpha(S\cap T)\neq 1$. We write $\boldsymbol{L}(\boldsymbol{M})$ for the class of all non-null subsets S of M such that S is orthogonal to every $X\in\boldsymbol{M}$, and $\boldsymbol{M}^*$ for the class of all minimal members of $\boldsymbol{L}(\boldsymbol{M})$.

(2.4) *If $(X,Y)\in\boldsymbol{L}(\boldsymbol{M})$, $a\in X\cap Y$ and $b\in X-(X\cap Y)$, then there exists $Z\in\boldsymbol{L}(\boldsymbol{M})$ such that $b\in Z\subseteq(X\cup Y)-\{a\}$.*

Proof. Assume the theorem false for some X, Y, a, b. We construct a sequence $(a_0,a_1,\cdots,a_k)$ of cells of $X\cup Y$ and a sequence $(T_1,\cdots,T_k)$ of points of $\boldsymbol{M}$ as follows. We put $a_0=a$. If we have determined the cells a_i as far as a_r, and b is not among them, we take T_{r+1} to be any point of $\boldsymbol{M}$ such that $T_{r+1}\cap((X\cup Y)-\{a_0,\cdots,a_r\})$ has just one cell, c say. This is possible since $(X\cup Y)-\{a_0,\cdots,a_r\}\notin L(M)$, by our assumption. We then write $a_{r+1}=c$. This construction terminates with the first a_k identical with b.

There exists $U\in\boldsymbol{M}$ such that $b\in U\cap(X\cup Y)\subseteq\{a_0,\cdots,a_k\}$, for example $U=T_k$. For each such U let $p(U)$ be the greatest $j<k$ such that $a_j\in U$. Such a j exists since $\alpha(U\cap X)\neq 1$. If $p(U)>0$ we can apply Axiom II to U and $T_{p(U)}$ to prove the existence of $U'\in\boldsymbol{M}$ such that $b\in U'\cap(X\cup Y)\subseteq\{a_0,\cdots,a_k\}$ and $p(U')<p(U)$. Hence we can choose U so that $p(U)=0$. But then $Y\notin L(\boldsymbol{M})$.

(2.5) *$\boldsymbol{M}^*$ is a matroid on M.*

Proof. By its definition $\boldsymbol{M}^*$ satisfies Axiom I. Suppose $(X,Y)\in\boldsymbol{M}^*$, $a\in X\cap Y$ and $b\in X-(X\cap Y)$. By (2.4) there exists $Z\in\boldsymbol{L}(\boldsymbol{M})$ such that $b\in Z\subseteq(X\cup Y)-\{a\}$. Choose such a Z so that $\alpha(Z)$ is as small as possible. If $Z\notin\boldsymbol{M}^*$ there exists $T\in\boldsymbol{L}(\boldsymbol{M})$ such that $T\subseteq Z-\{b\}$. Choose $c\in T$. By (2.4) there exists $Z'\in\boldsymbol{L}(\boldsymbol{M})$ such that $b\in Z'\subseteq(Z\cup T)-\{c\}$, which contradicts the definition of Z. Hence $Z\in\boldsymbol{M}^*$. Thus $\boldsymbol{M}^*$ satisfies Axiom II.

We call $\boldsymbol{M}^*$ the *dual matroid* of $\boldsymbol{M}$.

If D is a dendroid of $\boldsymbol{M}$ and $b\in M-D$ we define $K(\boldsymbol{M},D,b)$ as the subset of M consisting of b and each $a\in D$ such that $b\in J(\boldsymbol{M},D,a)$.

(2.6) $K(\boldsymbol{M}, D, b) \in \boldsymbol{M}^*$.

Proof. If $K(\boldsymbol{M}, D, b) \notin \boldsymbol{L}(\boldsymbol{M})$ choose $X \in \boldsymbol{M}$ such that $\alpha(X \cap K(\boldsymbol{M}, D, b)) = 1$ and $\alpha(X \cap D)$ has the least value consistent with this. Write $X \cap K(\boldsymbol{M}, D, b) = \{c\}$. If e belongs to $D \cap X$ but not to $K(\boldsymbol{M}, D, b)$ we have $b \notin J(\boldsymbol{M}, D, e)$ and so $c \notin J(\boldsymbol{M}, D, e)$. By Axiom II there exists $X' \in \boldsymbol{M}$ such that $c \in X' \subseteq (X \cup J(\boldsymbol{M}, D, e)) - \{e\}$. But then $\alpha(X' \cap K(\boldsymbol{M}, D, b)) = 1$ and $\alpha(X' \cap D) < \alpha(X \cap D)$, contrary to the definition of X. Since $X \cap D \neq \emptyset$ we deduce that $c \in D$ and $X = J(\boldsymbol{M}, D, c)$. But $c \in K(\boldsymbol{M}, D, b)$. Hence $b \in J(\boldsymbol{M}, D, c) = X$ and $\alpha(X \cap K(\boldsymbol{M}, D, b)) \geqq 2$, contrary to the definition of X. We deduce that $K(\boldsymbol{M}, D, b) \in \boldsymbol{L}(\boldsymbol{M})$. As no non-null proper subset of $K(\boldsymbol{M}, D, b)$ is orthogonal to all the sets $J(\boldsymbol{M}, D, a)$ it follows that $K(\boldsymbol{M}, D, b) \in \boldsymbol{M}^*$.

(2.7) *The dendroids of* $\boldsymbol{M}^*$ *are the complements in* M *of the dendroids of* $\boldsymbol{M}$.

Proof. Let D be any dendroid of $\boldsymbol{M}$. Then $M - D$ meets each $Y \in \boldsymbol{M}^*$, for no non-null subset of D is orthogonal to all the sets $J(\boldsymbol{M}, D, a)$. But no proper subset of $M - D$ meets all the sets $K(\boldsymbol{M}, D, b)$. Hence $M - D$ is a dendroid of $\boldsymbol{M}^*$, by (2.6).

Conversely suppose D' is a dendroid of $\boldsymbol{M}^*$. Then $M - D'$ meets each $X \in \boldsymbol{M}$, for no non-null subset of D' is orthogonal to all the sets $J(\boldsymbol{M}^*, D', a)$. Hence $\boldsymbol{M}$ has a dendroid D such that $D \subseteq M - D'$, that is $D' \subseteq M - D$. But $M - D$ is a dendroid of $\boldsymbol{M}^*$ by the result of the preceding paragraph. Hence $D' = M - D$.

Corollary I. $d\boldsymbol{M} + d\boldsymbol{M}^* = \alpha(M) - 2$ (by (2.2)).

Corollary II. $K(\boldsymbol{M}, D, b) = J(\boldsymbol{M}^*, M - D, b)$ (by (2.6)).

$$(\boldsymbol{M}^*)^* = \boldsymbol{M}. \tag{2.8}$$

Proof. By (2.7) $\boldsymbol{M}$ and $(\boldsymbol{M}^*)^*$ have the same dendroids. Hence they are identical, by (2.3).

For each $S \subseteq M$ we have the following identities.

$$\boldsymbol{M}^* \times S = (\boldsymbol{M} \cdot S)^*, \tag{2.9}$$

$$\boldsymbol{M}^* \cdot S = (\boldsymbol{M} \times S)^*. \tag{2.10}$$

The notation is that of HII, §3. To prove (2.9) suppose $X \in \boldsymbol{M}^* \times S$. Then $X \in \boldsymbol{M}^*$, that is X is orthogonal to each $Y \in \boldsymbol{M}$. Hence X is orthogonal to the intersection with S of each $Y \in \boldsymbol{M}$ and therefore $X \in \boldsymbol{L}(\boldsymbol{M} \cdot S)$. Accordingly there exists $X' \in (\boldsymbol{M} \cdot S)^*$ such that $X' \subseteq X$. Conversely suppose $X' \in (\boldsymbol{M} \cdot S)^*$. Then X' is orthogonal to each member of $\boldsymbol{M} \cdot S$, and therefore to each $Y \in \boldsymbol{M}$, by HII, (3.1). Hence $X' \in \boldsymbol{L}(\boldsymbol{M})$ and there exists $X \in \boldsymbol{M}^* \times S$ such that $X \subseteq X'$. Formula (2.9) now follows by Axiom I. To prove (2.10) we write $\boldsymbol{M}^*$ for $\boldsymbol{M}$ in (2.9), take dual matroids and use (2.8).

Let R denote either the ring of integers or the ring of residues modulo a prime. Let N be a chain-group on M over R, as defined in HII, §1. The dual chain-group N^* of N clearly has the property that $\alpha(|f| \cap |g|) \neq 1$ for any

$f \in N$ and $g \in N^*$. Hence if $X \in \boldsymbol{M}(N^*)$ there exists $Y \in (\boldsymbol{M}(N))^*$ such that $Y \subseteq X$.

If $Y \in (\boldsymbol{M}(N))^*$ we can write $Y = K(\boldsymbol{M}(N), D, b)$, for suitable D and b, by (2.1) and (2.7), Corollary II. We can find $d\boldsymbol{M}(N)+1$ linearly independent chains f_a $(a \in D)$ of N such that $|f_a| = J(\boldsymbol{M}(N), D, a)$, by (2.2). We define a chain g on M over R such that $|g| = Y$ according to the following rules.

$$g(b) = -\prod_{a \in D} f_a(a),$$

$$g(c) = f_c(b) \prod_{a \in D - \{c\}} f_a(a), \qquad (c \in D).$$

It is easily verified that

$$\sum_{e \in M} f(e) g(e) = 0$$

if f is any of the chains f_a. Hence the relation is true for each $f \in N$, since each such chain is linearly dependent on the f_a, by HII, (2.4). Hence $g \in N^*$. Accordingly there exists $X \in \boldsymbol{M}(N^*)$ such that $X \subseteq Y$.

Applying Axiom I to the above results we obtain

(2.11) $$\boldsymbol{M}(N^*) = (\boldsymbol{M}(N))^*.$$

(2.12) *The dual of a binary (regular) matroid is binary (regular).*

The terminology here is that of HII, §1. The theorem is a consequence of (2.11) and A, (5.1).

3. **Minors.** Let $\boldsymbol{M}$ be any matroid on a set M. Then if $T \subseteq S \subseteq M$ we have the following identities.

(3.1) $$(\boldsymbol{M} \times S) \times T = \boldsymbol{M} \times T,$$

(3.2) $$(\boldsymbol{M} \cdot S) \cdot T = \boldsymbol{M} \cdot T,$$

(3.3) $$(\boldsymbol{M} \cdot S) \times T = (\boldsymbol{M} \times (M - (S - T))) \cdot T,$$

(3.4) $$(\boldsymbol{M} \times S) \cdot T = (\boldsymbol{M} \cdot (M - (S - T))) \times T.$$

Formula (3.1) follows at once from the definition of $\boldsymbol{M} \times S$ in HII, §3. To prove (3.2) we write $\boldsymbol{M}^*$ for $\boldsymbol{M}$ in (3.1), apply (2.9) and take dual matroids.

To prove (3.3) suppose $X \in (\boldsymbol{M} \cdot S) \times T$. Then there exists $X' \in \boldsymbol{M}$ such that $X' \cap (S-T) = \emptyset$ and $X' \cap T = X$. But then $X' \in \boldsymbol{M} \times (M-(S-T))$ and there exists $Y \in (\boldsymbol{M} \times (M-(S-T))) \cdot T$ such that $Y \subseteq X' \cap T = X$. Conversely suppose $Y \in (\boldsymbol{M} \times (M-(S-T))) \cdot T$. Then there exists $Y' \in \boldsymbol{M}$ such that $Y' \cap (S-T) = \emptyset$ and $Y' \cap T = Y$. Hence there exists $X \in (\boldsymbol{M} \cdot S) \times T$ such that $X \subseteq Y' \cap T = Y$. Now (3.3) follows by Axiom I.

We obtain (3.4) by writing $M-(S-T)$ for S in (3.3).

We refer to the matroids of the form $(\boldsymbol{M} \times S) \cdot T$ as the *minors* of $\boldsymbol{M}$. By (3.3) and (3.4) they are also the matroids of the form $(\boldsymbol{M} \cdot S) \times T$. We note that $\boldsymbol{M}$, $\boldsymbol{M}\cdot S$ and $\boldsymbol{M}\times S$ are minors of $\boldsymbol{M}$, for $\boldsymbol{M}=(\boldsymbol{M}\times M)\cdot M$, $\boldsymbol{M}\cdot S=(\boldsymbol{M}\times M)\cdot S$ and $\boldsymbol{M}\times S=(\boldsymbol{M}\times S)\cdot S$.

(3.5) *Every minor of a minor of* $\boldsymbol{M}$ *is a minor of* $\boldsymbol{M}$.

(3.6) *The minors of* $\boldsymbol{M}^*$ *are the dual matroids of the minors of* $\boldsymbol{M}$.

(3.7) *The minors of a binary (regular) matroid are binary (regular).*

(3.5) is a consequence of the identities (3.1)–(3.4), and (3.6) is implied by (2.9) and (2.10). (3.7) follows from HII, (4.3).

A subset S of M is a *separator* of $\boldsymbol{M}$ if no $X\in\boldsymbol{M}$ meets both S and $M-S$. We call $\boldsymbol{M}$ *connected* if it has no separator other than the trivial ones $\emptyset$ and M. We have at once:

(3.8) *If* S *is a separator of* $\boldsymbol{M}$ *and* $U\subseteq M$, *then* $S\cap U$ *is a separator of both* $\boldsymbol{M}\times U$ *and* $\boldsymbol{M}\cdot U$.

(3.9) *A subset* S *of* M *is a separator of* $\boldsymbol{M}$ *if and only if* $\boldsymbol{M}\times S=\boldsymbol{M}\cdot S$.

Proof. Suppose S is a separator of $\boldsymbol{M}$. If $X\in\boldsymbol{M}\cdot S$ there exists $X'\in\boldsymbol{M}$ such that $X=X'\cap S$. Then $X'=X$ since S is a separator, and so $X\in\boldsymbol{M}\times S$. Conversely suppose $X\in\boldsymbol{M}\times S$. Then $X\in\boldsymbol{M}$ and there exists $Y\in\boldsymbol{M}\cdot S$ such that $Y\subseteq X$. By the result just proved $Y\in\boldsymbol{M}\times S$ and therefore $X=Y\in\boldsymbol{M}\cdot S$, by Axiom I.

Next suppose S is not a separator of $\boldsymbol{M}$. Then there exists $X\in\boldsymbol{M}$ such that $X\cap S$ and $X\cap(M-S)$ are both non-null. There exists $Y\in\boldsymbol{M}\cdot S$ such that $Y\subseteq X\cap S$. But then $Y\notin\boldsymbol{M}$, by Axiom I, and so $Y\notin\boldsymbol{M}\times S$. Hence $\boldsymbol{M}\times S\neq\boldsymbol{M}\cdot S$.

We define an *elementary* separator of $\boldsymbol{M}$ as a minimal non-null separator of $\boldsymbol{M}$. It is clear that if $M\neq\emptyset$ the elementary separators of $\boldsymbol{M}$ are disjoint and have M as their union. An elementary separator may consist of a single cell a. This happens when either $\{a\}\in\boldsymbol{M}$ or no point of $\boldsymbol{M}$ includes a. An elementary separator of $\boldsymbol{M}$ having more than one cell is necessarily a connected flat of $\boldsymbol{M}$, as defined in HI, §1.

(3.10) *The elementary separators of* $\boldsymbol{M}^*$ *are identical with those of* $\boldsymbol{M}$.

This follows from (2.9), (2.10) and (3.9).

4. **Circuits and bonds.** For convenience we state here some of the fundamental definitions of graph theory which were assumed in A.

A *graph* G is defined by a set $E(G)$ of *edges*, a set $V(G)$ of *vertices*, and a relation of *incidence* which associates with each edge a pair of vertices, not necessarily distinct, called its *ends*. An edge is a *link* or *loop* according as its ends are distinct or coincident. A vertex which is not an end of any edge is called *isolated*.

In this paper we suppose $E(G)$ and $V(G)$ both finite. We denote the number of members of $E(G)$ and $V(G)$ by $\alpha_1(G)$ and $\alpha_0(G)$ respectively.

A graph H is a *subgraph* of G if $E(H)\subseteq E(G)$, $V(H)\subseteq V(G)$ and each $A\in E(H)$ has the same ends in H as in G. The *union* of given subgraphs $G_1,\cdots,G_k$ of G is the subgraph H of G such that $E(H)=\bigcup_i E(G_i)$ and $V(H)=\bigcup_i V(G_i)$. If $W\subseteq V(G)$ we write $G[W]$ for the subgraph of G whose vertices are the members of W and whose edges are those edges of G which have both ends in W.

A sequence $P=(a_0, A_1, a_1, \cdots, A_n, a_n)$, having at least one term, is a *path* in G from a_0 to a_n if the following conditions are satisfied.

(i) The terms of P are alternately vertices a_i and edges A_j of G.

(ii) If $1 \leqq j \leqq n$ then a_{j-1} and a_j are the two ends in G of A_j.

We call P *degenerate* if it has only one term, *simple* if the a_i are all distinct, and *re-entrant* if $a_0=a_n$. We call it *circular* if it is re-entrant, nondegenerate and such that the vertices $a_0, \cdots, a_{n-1}$ are all distinct.

Clearly if there is any path in G from a_0 to a_n there is a simple path in G from a_0 to a_n.

If $(x, y) \in V(G)$ we say x and y are *connected* in G if there is a path in G from x to y. The relation of connection is clearly an equivalence relation. Hence if $V(G)$ is non-null it can be partitioned into disjoint non-null subsets $V_1, \cdots, V_k$ such that two vertices are connected in G if and only if they belong to the same set V_i. The subgraphs $G[V_i]$ are the *components* of G. No two of them have an edge or vertex in common and their union is G. We write $p_0(G)$ for the number of components of G. We call G a *connected* graph if $p_0(G)=0$ or 1. The former case arises only when $E(G)$ and $V(G)$ are both null. Clearly each component of a graph is connected.

If $S \subseteq E(G)$ we define the graphs $G \cdot S$, $G:S$ and $G \times S$ as in A, §2.

Bonds and circuits are defined in A, §2. We write $\boldsymbol{C}(G)$ for the class of all subsets S of $E(G)$ such that $G \cdot S$ is a circuit of G, and $\boldsymbol{B}(G)$ for the class of all $S \subseteq E(G)$ such that $G \times S$ is a bond of G. Thus $S \in \boldsymbol{C}(G)$ if and only if S is the set of edges of some circular path in G, and $S \in \boldsymbol{B}(G)$ if and only if there are distinct components K and L of G: $(E(G)-S)$ such that each $A \in S$ has one end (in G) a vertex of K and the other a vertex of L.

(4.1) *$\boldsymbol{B}(G)$ and $\boldsymbol{C}(G)$ are regular matroids on $E(G)$.*

(4.2) *$\boldsymbol{B}(G)=(\boldsymbol{C}(G))^*$.*

These theorems can be proved from the results of A. In A, §2 it was shown that there are two regular chain-groups $\Delta(G)$ and $\Gamma(G)$ on $E(G)$ associated with G (taken with a fixed orientation). By A, (2.4) and (2.7), $\boldsymbol{B}(G)$ and $\boldsymbol{C}(G)$ are the matroids of $\Delta(G)$ and $\Gamma(G)$ respectively. This proves (4.1). By A, (5.5), $\Delta(G)$ and $\Gamma(G)$ are dual chain-groups. Hence (4.2) follows by (2.11).

$$(4.3) \qquad d\boldsymbol{B}(G) = \alpha_0(G) - p_0(G) - 1,$$

$$(4.4) \qquad d\boldsymbol{C}(G) = \alpha_1(G) - \alpha_0(G) + p_0(G) - 1.$$

To prove (4.3) we observe that by formula (2.2) of A the coboundary of a 0-chain f on G is zero if and only if in each component of G all the vertices have the same coefficient in f. This implies that the maximum number of linearly independent chains of $\Delta(G)$ is $\alpha_0(G)-p_0(G)$. (4.3) now follows by HII, (2.4). We derive (4.4) from (4.3) by using (4.2) and (2.7), Corollary I.

If T and U are complementary subsets of $V(G)$ we write $Q(T, U)$ for the set of all edges of G having one end in T and the other in U. We note that each $S \in \mathbf{B}(G)$ is of the form $Q(T, U)$.

(4.5) *If $A \in Q(T, U)$ there exists $S \in \mathbf{B}(G)$ such that $A \in S \subseteq Q(T, U)$.*

Proof. We can find S such that $A \in S \subseteq Q(T, U)$, S is of the form $Q(T', U')$, and $\alpha(S)$ has the least value consistent with these conditions. Let H be the component of $G[T']$ having an end of A in G as a vertex. Then H is the only component of $G[T']$ having an end of a member of S as a vertex, for otherwise we would have $A \in Q(V(H), V(G) - V(H)) \subset S$, contrary to the definition of S. Similarly only one component of $G[U']$ has an end of a member of S as a vertex. It follows that $S \in \mathbf{B}(G)$.

If $S \subseteq E(G)$ we have the following identities.

(4.6) $$\mathbf{C}(G \cdot S) = \mathbf{C}(G) \times S,$$

(4.7) $$\mathbf{C}(G \times S) = \mathbf{C}(G) \cdot S,$$

(4.8) $$\mathbf{B}(G \cdot S) = \mathbf{B}(G) \cdot S,$$

(4.9) $$\mathbf{B}(G \times S) = \mathbf{B}(G) \times S.$$

Formula (4.6) follows immediately from the definitions. To prove (4.7) suppose $Y \in \mathbf{C}(G \times S)$. There is a circular path $P = (C_0, A_1, C_1, \cdots, A_k, C_0)$ in $G \times S$ whose edges A_i are the members of Y. We recall that the vertices of $G \times S$ are components of the graph $G: (E(G) - S)$. For $1 \leqq j \leqq k$ let x_j and y_j be ends in G of A_j, distinct if possible, belonging to C_{j-1} and C_j respectively ($C_k = C_0$). In P we replace the first term by x_1, the last by a simple path in the graph C_0 from y_k to x_1, and each intermediate C_i by a simple path in C_i from y_i to x_{i+1}. There results a circular path in G. Hence there exists $T \in \mathbf{C}(G)$ such that $T \cap S = Y$. Accordingly there exists $Z \in \mathbf{C}(G) \cdot S$ such that $Z \subseteq Y$. Conversely suppose $Z \in \mathbf{C}(G) \cdot S$. Then there is a circular path $Q = (a_0, A_1, a_1, \cdots, A_n, a_0)$ in G such that those edges A_i which belong to S are the members of Z. In Q we delete each edge not in Z and its succeeding vertex-term. We then replace each remaining term a_i by the component C_i of $G: (E(G) - S)$ to which it belongs. There results a re-entrant path Q' in $G \times S$ whose edges are the members of Z. This must contain a circular path in $G \times S$ as a subsequence. Hence there exists $Y \in \mathbf{C}(G \times S)$ such that $Y \subseteq Z$. Formula (4.7) now follows by Axiom I.

To prove (4.8) and (4.9) we take dual matroids in (4.6) and (4.7), apply (2.9) and (2.10), and then use (4.2).

We call a matroid $\mathbf{M}$ *graphic* or *cographic* if there is a graph G such that $\mathbf{M} = \mathbf{B}(G)$ or $\mathbf{M} = \mathbf{C}(G)$ respectively. By the four identities just proved we have:

(4.10) *The minors of a graphic (cographic) matroid are graphic (cographic).*

A *complete* 5-*graph* is a graph having just five vertices $a_1, \cdots, a_5$ and just ten edges L_{ij} $(1 \leqq i < j \leqq 5)$, the ends of L_{ij} being a_i and a_j. A *Thomsen graph* is a graph having just six vertices $a_1, a_2, a_3, b_1, b_2, b_3$ and just nine edges

L_{ij} $(1 \leqq (i, j) \leqq 3)$ such that the ends of L_{ij} are a_i and b_j. Diagrams of these graphs are given in Figure I, the Thomsen graph being shown on the right. We refer to the graphs of these two kinds as the *Kuratowski graphs*.

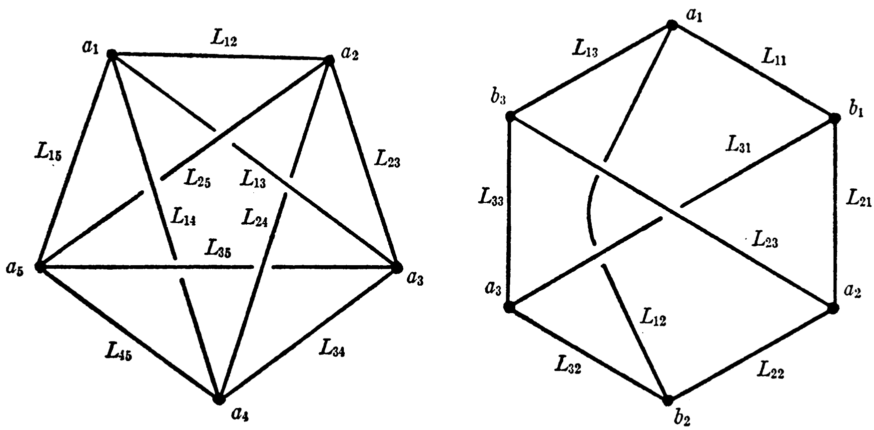

FIG. I

We can now give a more precise statement of the main theorem of the paper. The *bond-matroid* and *circuit-matroid* of G mentioned in the Introduction are $\mathbf{B}(G)$ and $\mathbf{C}(G)$ respectively. The word "contain" used there means "have as a minor."

THEOREM. *A matroid* $\mathbf{M}$ *is graphic (cographic) if and only if it is regular and has no minor which is the circuit-matroid (bond-matroid) of a Kuratowski graph.*

We complete the proof of this theorem in §8.

Following Hassler Whitney [5] we say G is *separable* if it is not connected or if there are complementary non-null subsets S and T of $E(G)$ such that $G \cdot S$ and $G \cdot T$ have only one vertex in common.

A subgraph $G \cdot S$ of G is a *separate* of G if it is non-null, non-separable, and such that no component of $G:(E(G)-S)$ includes more than one vertex of $G \cdot S$. (Separates are called "components" in [5]). By this definition a separate must have at least one edge.

(4.11) *Suppose* $S \subseteq E(G)$. *Then* $G \cdot S$ *is a separate of* G *if and only if* S *is an elementary separator of* $\mathbf{B}(G)$.

Proof. Suppose S is an elementary separator of $\mathbf{B}(G)$. Then $G \cdot S$ is non-null. Assume $G \cdot S$ separable. Then there are complementary non-null subsets U and V of S such that $G \cdot U$ and $G \cdot V$ have at most one vertex in common. For any $Y \in \mathbf{B}(G) \times S$ we have $Y \in \mathbf{B}(G) \cdot S = \mathbf{B}(G \cdot S)$, by (3.9) and (4.8). Hence there are distinct components H and K of $(G \cdot S):(S-Y)$ such that each $A \in Y$ has one end in H and the other in K. Without loss of generality we may suppose H includes no common vertex of $G\cdot U$ and $G\cdot V$. It follows that H is a subgraph of one of these and hence that Y is a subset of U or V. Thus U is a separator of $\boldsymbol{B}(G)\times S$ and therefore of $\boldsymbol{B}(G)$, contrary to the definition of S. We deduce that $G\cdot S$ is nonseparable.

Assume $G:(E(G)-S)$ has a component L which includes two distinct vertices x and y of $G\cdot S$. We can find a simple path P from x to y in $G\cdot S$, and a simple path Q from y to x in L. By a proper choice of y and P we can arrange that no vertex in P other than x and y is a vertex of L. Combining P and Q to form a circular path in G we see that there exists $X\in\boldsymbol{C}(G)$ meeting both S and $E(G)-S$. Hence S is not an elementary separator of $\boldsymbol{C}(G)$. This is contrary to the definition of S, by (3.10) and (4.2). We conclude that $G\cdot S$ is a separate of G.

Conversely suppose $G\cdot S$ is a separate of G. Choose $A\in S$ and let T be the elementary separator of $\boldsymbol{B}(G)$ containing A. Then $G\cdot T$ is a separate of G by the preceding argument.

Each component of $(G\cdot S):(S-(S\cap T))$ is a subgraph of a component of $G:(E(G)-T)$. It therefore has at most one vertex in common with the subgraph $G\cdot(S\cap T)$ of $G\cdot T$. Since $G\cdot S$ is nonseparable it follows that no component of $(G\cdot S):(S-(S\cap T))$ has an edge, that is $(S\cap T)=S$. A similar argument, with S and T interchanged, shows that $(S\cap T)=T$. Hence S is the elementary separator T of $\boldsymbol{B}(G)$.

If $S\subseteq E(G)$ we call the common vertices of $G\cdot S$ and $G\cdot(E(G)-S)$ the *vertices of attachment* of S, and their number the *attachment-number* $w(S)$ of S in G. In a nonseparable graph no non-null proper subset of $E(G)$ has attachment-number 0 or 1.

Suppose $S\subseteq E(G)$ and $w(S)=2$. Let the vertices of attachment of S be x and y. We construct a graph G' such that $E(G')=E(G)$, $V(G')=V(G)$ and the same incidence relations hold as in G, save only that the incidence with x and y of members of S is governed by the following rule: if $A\in S$ then A is incident with x (y) in G' if and only if it is incident with y (x) in G. We say that G' is obtained from G by *reversing* S. We can recover G from G' by reversing S again.

Suppose $Y\in\boldsymbol{B}(G)$. Then $Y=Q(T,\ U)$ for suitable complementary subsets T and U of $V(G)$. If x and y belong to the same set T or U we write $T'=T$ and $U'=U$. But if x and y belong to different sets T and U we write $T'=(T\cap V(G\cdot(E(G)-S)))\cup(U\cap(V(G\cdot S)-\{x,y\}))$ and $U'=(U\cap V(G\cdot(E(G)-S)))\cup(T\cap(V(G\cdot S)-\{x,\ y\}))$. It is readily verified that in each case $Y=Q(U',\ V')$ in G'. Hence there exists $Y'\in\boldsymbol{B}(G')$ such that $Y'\subseteq Y$, by (4.5). Similarly for each $Y'\in\boldsymbol{B}(G')$ there exists $Y\in\boldsymbol{B}(G)$ such that $Y\subseteq Y'$. Hence $\boldsymbol{B}(G')=\boldsymbol{B}(G)$. We thus have:

(4.12) *$\boldsymbol{B}(G)$ is invariant under the operation of reversing a subset S of $E(G)$ such that $w(S)=2$.*

(4.13) *If $Y\in\boldsymbol{B}(G)$ and $\boldsymbol{B}(G)\cdot(E(G)-Y)$ is connected there exists $a\in V(G)$ such that $Y=Q(\{a\},\ V(G)-\{a\})$.*

Proof. $G\cdot(E(G)-Y)$ is nonseparable and therefore connected, by (4.8) and (4.11). It is thus a component of $G:(E(G)-Y)$. The other component of this graph including an end of each $A\in Y$ thus has no edge and only one vertex, a say.

5. Convex subclasses of a matroid. The results on convex subclasses of a matroid $\boldsymbol{M}$ given in HI, §5 need to be supplemented by the following theorems.

(5.1) *Let* $\mathbf{C}$ *and* $\mathbf{C}'$ *be convex subclasses of* $\boldsymbol{M}$ *and let* S *be a connected flat of* $\boldsymbol{M}$ *having a point of* $\boldsymbol{M}-\mathbf{C}$ *and a point of* $\boldsymbol{M}-\mathbf{C}'$ *as subsets. Then there exists a point of* $(\boldsymbol{M}-\mathbf{C})\cap(\boldsymbol{M}-\mathbf{C}')$ *on* S.

Proof. Assume the theorem false. There are points X and Y of $\boldsymbol{M}$ on S such that $X\in\mathbf{C}\cap(\boldsymbol{M}-\mathbf{C}')$ and $Y\in\mathbf{C}'\cap(\boldsymbol{M}-\mathbf{C})$. By HI, (5.1), there exists $Z\in\boldsymbol{M}$ on S such that $X\cup Z$ is a connected line of $\boldsymbol{M}$ and $Z\in\boldsymbol{M}-\mathbf{C}$. By assumption $Z\in\mathbf{C}'$. But there is a point of $\boldsymbol{M}$ other than X and Z on $X\cup Z$, by HI, (3.2). By the definition of a convex subclass this point must belong to both $\boldsymbol{M}-\mathbf{C}$ and $\boldsymbol{M}-\mathbf{C}'$.

(5.2) *If* $\boldsymbol{M}$ *is binary and* S *is any set of cells of* $\boldsymbol{M}$ *then the class* $\mathbf{C}$ *of all* $X\in\boldsymbol{M}$ *such that* $S\subseteq X$ *or* $X\cap S=\emptyset$ *is a convex subclass of* $\boldsymbol{M}$.

Proof. Let X, Y and Z be the three points on some connected line L of $\boldsymbol{M}$. (HII, (2.7)). Suppose $(X, Y)\in\mathbf{C}$. Then $Z\in\mathbf{C}$, for Z is the mod 2 sum of X and Y by HII, (2.6) and $X\cap Y\neq\phi$ by HI, (3.1).

6. Properties of a matroid at a point. In this section and the next we study the relation of a point Y of a matroid $\boldsymbol{M}$ on a set M to the rest of the matroid.

$$(6.1)\qquad d(\boldsymbol{M}\cdot(M-Y)) = d\boldsymbol{M}-1.$$

Proof. Choose $a\in Y$ and let D be a dendroid of $\boldsymbol{M}\cdot(M-Y)$. Then $D\cup\{a\}$ meets each $X\in\boldsymbol{M}$. For each $b\in D$ there exists $Z_b\in\boldsymbol{M}\cdot(M-Y)$ such that $Z_b\cap D=\{b\}$. Hence there exists also $X_b\in\boldsymbol{M}$ such that $X_b\cap D=\{b\}$. By Axiom II we can arrange that $a\notin X_b$. Hence no proper subset of $D\cup\{a\}$ meets each $X\in\boldsymbol{M}$ and so $D\cup\{a\}$ is a dendroid of $\boldsymbol{M}$. The theorem follows by (2.2).

(6.2) *Suppose* $S\subseteq M-Y$ *and* d *is an integer* $\geqq -1$. *Then* S *is a* d*-flat of* $\boldsymbol{M}\cdot(M-Y)$ *if and only if* $S\cup Y$ *is a* $(d+1)$*-flat of* $\boldsymbol{M}$.

Proof. It is clear that if S is a flat of $\boldsymbol{M}\cdot(M-Y)$ then $S\cup Y$ is a flat of $\boldsymbol{M}$. The converse is also true, by HII, (3.1). To complete the proof we observe that $d((\boldsymbol{M}\cdot(M-Y))\times S)=d((\boldsymbol{M}\times(S\cup Y))\cdot S)=d(\boldsymbol{M}\times(S\cup Y))-1$, by (3.3) and (6.1).

(6.3) *Let* S *be a connected flat of* $\boldsymbol{M}\cdot(M-Y)$. *Then either* $S\cup Y$ *is a connected flat of* $\boldsymbol{M}$ *or* $(\boldsymbol{M}\cdot(M-Y))\times S=\boldsymbol{M}\times S$.

Proof. Since S is connected in $\boldsymbol{M}\cdot(M-Y)$ the only possible separation of the flat $S\cup Y$ of $\boldsymbol{M}$ is $\{S, Y\}$ (HI, §4). If such a separation exists we have $(\boldsymbol{M}\cdot(M-Y))\times S=(\boldsymbol{M}\times(S\cup Y))\cdot S=(\boldsymbol{M}\times(S\cup Y))\times S=\boldsymbol{M}\times S$, by (3.1), (3.3) and (3.9).

From now on we suppose $\boldsymbol{M}$ binary. If $Z \in \boldsymbol{M}\cdot(M-Y)$ then $Y \cup Z$ is a line of $\boldsymbol{M}$, by (6.2). If it is connected it has just two points other than Z, by HII, (2.7). Their intersections with Y are complementary non-null subsets T and U of Y, by HII, (2.6). We call the unordered pair $\{T, U\}$ the *partition* of Y determined by Z. If $Y \cup Z$ is not connected its two points (HI, (3.2)) must be Y and Z. In this case we define the partition of Y determined by Z as the unordered pair $\{Y, \varnothing\}$.

We say that Z *cuts* a subset S of Y if it determines a partition $\{T, U\}$ of Y such that $\varnothing \subset S \cap T \subset S$.

(6.4) *If $\boldsymbol{M}$ is binary the points of $\boldsymbol{M}\cdot(M-Y)$ which do not cut a given subset S of Y constitute a convex subclass of $\boldsymbol{M}\cdot(M-Y)$.*

Proof. Let Y_1, Y_2 and Y_3 be the three points on a connected line L of $\boldsymbol{M}\cdot(M-Y)$. Suppose Y_1 and Y_2 do not cut S. Then each point of $\boldsymbol{M}$ on the lines $Y \cup Y_1$ and $Y \cup Y_2$ of $\boldsymbol{M}$ either contains S or does not meet S. If X_3 is a point on the line $Y \cup Y_3$ of $\boldsymbol{M}$ other than Y there is a line L' of $\boldsymbol{M}$ on X_3 and the plane $L \cup Y$ which is not on Y, by HI, (2.4). This meets $Y \cup Y_1$ and $Y \cup Y_2$ in distinct points, by HI, (2.2) and (2.5). Hence X_3 either contains S or does not meet S, by (5.2). It follows that Y_3 does not cut S.

We refer to the elementary separators of $\boldsymbol{M}\cdot(M-Y)$ as the *bridges* of Y in $\boldsymbol{M}$. A bridge of Y having only one cell is *unicellular*. If $\boldsymbol{M}=\boldsymbol{B}(G)$ the bridges of Y correspond to the separates of $G\cdot(E(G)-Y)$, by (4.8) and (4.11). If $\boldsymbol{M}=\mathbf{C}(G)$ they correspond to the separates of $G\times(E(G)-Y)$, by (3.10), (4.2), (4.7) and (4.11). In the Thomsen graph of Figure I for example $\{L_{12}\}$ is a unicellular bridge of the point $\{L_{11}, L_{21}, L_{22}, L_{32}, L_{33}, L_{13}\}$ of $\mathbf{C}(G)$. It determines the partition $\{\{L_{11}, L_{21}, L_{22}\}, \{L_{32}, L_{33}, L_{13}\}\}$.

Two points Z and Z' of $\boldsymbol{M}\cdot(M-Y)$ are *skew* with respect to Y if they determine partitions $\{T, U\}$ and $\{T', U'\}$ of Y such that the intersections $T \cap T'$, $T \cap U'$, $U \cap T'$ and $U \cap U'$ are all non-null. Two bridges B and B' of Y in $\boldsymbol{M}$ are *skew* if there are points Z and Z' of $\boldsymbol{M}\cdot(M-Y)$, skew with respect to Y, such that $Z \subseteq B$ and $Z' \subseteq B'$. We also say that a point Z of $\boldsymbol{M}\cdot(M-Y)$ is *skew* to a bridge B of Y in $\boldsymbol{M}$ if there is a point Z' of $\boldsymbol{M}\cdot(M-Y)$ on B which is skew to Z.

(6.5) *Let B_1, B_2 and B_3 be distinct bridges of Y in $\boldsymbol{M}$ such that B_2 is skew to both B_1 and B_3. Then either there is a point Z_2 of $\boldsymbol{M}\cdot(M-Y)$ on B_2 which is skew to both B_1 and B_3 or there are points Z_1 and Z_3 of $\boldsymbol{M}\cdot(M-Y)$ on B_1 and B_3 respectively such that Z_1, B_2 and Z_3 are mutually skew.*

Proof. There are points Z_1, Z_2', Z_2'' and Z_3 of $\boldsymbol{M}\cdot(M-Y)$ on B_1, B_2, B_2 and B_3 respectively such that Z_1 is skew to Z_2' and Z_2'' is skew to Z_3. If Z_1 and Z_3 are skew the second alternative of the theorem holds. In the remaining case Z_1 and Z_3 determine partitions $\{S_1, T_1\}$ and $\{S_3, T_3\}$ of Y such that $T_1 \cap T_3 = \varnothing$, i.e., $T_1 \subseteq S_3$ and $T_3 \subseteq S_1$.

Now Z_2' cuts T_1, Z_2'' cuts T_3 and B_2 is a connected flat of $\boldsymbol{M}\cdot(M-Y)$, Hence, by (5.1) and (6.4) there exists $Z_2 \in \boldsymbol{M}\cdot(M-Y)$ on B_2 cutting both T_1 and T_3. Then Z_2 is skew to both Z_1 and Z_3, and therefore to both B_1 and B_3.

7. **Regular matroids.** In HII, §4, we defined geometrical figures of *Types* BI *and* BII. In the main theorem of HII we showed that a binary matroid is regular if and only if it has no figure of Type BI or BII. We make use of this theorem in the proofs of (7.1) and (7.2).

We suppose given a regular matroid $\boldsymbol{M}$ on a set M, and some $Y \in \boldsymbol{M}$.

(7.1) *Let Z_1 and Z_2 be distinct points on a connected line L of $\boldsymbol{M}\cdot(M-Y)$. Then Z_1 and Z_2 are not skew with respect to Y.*

Proof. $L \cup Y$ is a plane P of $\boldsymbol{M}$, by (6.2). It includes three distinct lines $Y \cup Z_1$, $Y \cup Z_2$ and $Y \cup Z_3$ of $\boldsymbol{M}$ on Y, where Z_3 is the third point of $\boldsymbol{M}\cdot(M-Y)$ on L.

Assume Z_1 and Z_2 skew. They determine partitions $\{S_1, T_1\}$ and $\{S_2, T_2\}$ of Y respectively. There are cells $a \in S_1 \cap S_2$, $b \in S_1 \cap T_2$, $c \in T_1 \cap S_2$ and $d \in T_1 \cap T_2$. The flats $\langle P-\{a\}\rangle$, $\langle P-\{b\}\rangle$, $\langle P-\{c\}\rangle$ and $\langle P-\{d\}\rangle$ of $\boldsymbol{M}$ are lines on P which are not on Y, by HI, (2.3). It is easily seen that they are distinct. For example $\langle P-\{a\}\rangle$ is the only one which is on the points $Z_1 \cup T_1$ and $Z_2 \cup T_2$ of $\boldsymbol{M}$.

As there are seven distinct lines on P the matroid $\boldsymbol{M}$ has a figure of Type BI. This is impossible since $\boldsymbol{M}$ is regular.

(7.2) *Let Z_1, Z_2 and Z_3 be points of $\boldsymbol{M}\cdot(M-Y)$ on distinct bridges B_1, B_2 and B_3 respectively of Y in $\boldsymbol{M}$. Let Z_i determine the partition $\{S_i, T_i\}$ of Y $(i=1, 2, 3)$. Then if S_3 meets both $S_1 \cap S_2$ and $T_1 \cap T_2$ it contains one of the sets $S_1 \cap T_2$ and $S_2 \cap T_1$.*

Proof. If the theorem is false we can choose $K=\{a, b, c, d, e, f, g\} \subseteq M$ such that $a \in Z_1$, $b \in Z_2$, $c \in Z_3$, $d \in S_1 \cap S_2 \cap S_3$, $e \in T_1 \cap T_2 \cap S_3$, $f \in S_1 \cap T_2$, $g \in S_2 \cap T_1$, $f \notin S_3$, $g \notin S_3$.

In the matroid $\boldsymbol{M}\cdot(M-Y)$ the flats $Z_1 \cup Z_2$, $Z_2 \cup Z_3$ and $Z_3 \cup Z_1$ are lines, and $Z_1 \cup Z_2 \cup Z_3$ is a plane. This follows from the definition of dimension in HI, §1. Hence $Y \cup Z_1 \cup Z_2 \cup Z_3$ is a 3-flat E of $\boldsymbol{M}$, by (6.2). There are points of $\boldsymbol{M}$ on E whose intersections with K are $\{a, d, f\}$, $\{a, g, e\}$, $\{b, d, g\}$, $\{b, e, f\}$, $\{c, f, g\}$, $\{c, d, e\}$ and $\{a, b, c\}$. For example $(Z_1 \cup S_1) \cap K=\{a, d, f\}$, and similar verifications may be made for the next five intersections. As for $\{a, b, c\}$ we observe that $Z_1 \cup T_1 \subset Z_1 \cup T_1 \cup Z_2 \cup T_2 \subset Z_1 \cup T_1 \cup Z_2 \cup T_2 \cup Z_3 \cup T_3 \subset E$. Hence $Z_1 \cup T_1 \cup Z_2 \cup T_2$ is a line of $\boldsymbol{M}$, by HI, (2.2). Its points are $Z_1 \cup T_1$, $Z_2 \cup T_2$ and their mod 2 sum, Q say. $Q \cap K=\{a, b, f, g\}$. Further $Z_1 \cup T_1 \cup Z_2 \cup T_2 \cup Z_3 \cup T_3$ is a plane P of $\boldsymbol{M}$ on E. The line $\langle P-\{e\}\rangle$ of $\boldsymbol{M}$ is on the points $Z_3 \cup T_3$ and Q. As these distinct points have a common cell f their mod 2 sum, R say, is a third point of $\boldsymbol{M}$ on $\langle P-\{e\}\rangle$. But $R \cap K=\{a, b, c\}$.

Considering the seven intersections with K set out above, we see that each cell of K occurs in just three of them, and that no two have two cells in common. Hence given any three cells x, y and z of K we can find a point of $\boldsymbol{M}$ on E including x but not y or z. It follows that no three of the planes $\langle E-\{x\}\rangle$, $(x \in K)$, of $\boldsymbol{M}$ have a common line; we can find a point on any two of them which is not on the third. Hence these seven planes on E are distinct and define a figure of Type BII. This is impossible since $\boldsymbol{M}$ is regular.

If B is any bridge of Y in $\boldsymbol{M}$ we define $\pi(\boldsymbol{M}, B, Y)$ as the class of all minimal non-null subsets of Y which are intersections of points of $\boldsymbol{M}\times(B\cup Y)$.

(7.3) *The members of $\pi(\boldsymbol{M}, B, Y)$ are disjoint and their union is Y. Moreover if $W\in\pi(\boldsymbol{M}, B, Y)$ then either $W=Y$ or there exists $Z\in\boldsymbol{M}\cdot(M-Y)$ on B such that $W\cup Z\in\boldsymbol{M}\times(B\cup Y)$.*

Proof. Let a be any cell of Y. If $(\boldsymbol{M}\cdot(M-Y))\times B$ is non-null let U be a subset of Y such that $a\in U$, some $Z\in\boldsymbol{M}\cdot(M-Y)$ on B determines the partition $\{U, Y-U\}$ of Y, and $\alpha(U)$ has the least value consistent with these conditions. If $(\boldsymbol{M}\cdot(M-Y))\times B$ is null we write $U=Y$.

Assume some $W\in\pi(\boldsymbol{M}, B, Y)$ satisfies $\varnothing\subset W\cap U\subset U$. Then there exists $X\in\boldsymbol{M}\times(B\cup Y)$ such that $\varnothing\subset X\cap U\subset U$. Clearly $X\cup Y$ is a connected flat of $\boldsymbol{M}\times(B\cup Y)$. By (5.2) and HI, (5.1), we can find $X'\in\boldsymbol{M}\times(B\cup Y)$ on $X\cup Y$ such that $\varnothing\subset X'\cap U\subset U$ and $X'\cup Y$ is a connected line of $\boldsymbol{M}\times(B\cup Y)$. Then, by (6.2), $X'-(X'\cap Y)$ is a point of $\boldsymbol{M}\cdot(M-Y)$ on B. It determines the partition $\{X'\cap Y, Y-(X'\cap Y)\}$ of Y, and so cuts U. We note that $(\boldsymbol{M}\cdot(M-Y))\times B$ is non-null and so Z is defined. By (6.4) and HI, (5.1), we can find $Z'\in\boldsymbol{M}\cdot(M-Y)$ on B which cuts U and is such that $Z\cup Z'$ is a connected line of $\boldsymbol{M}\cdot(M-Y)$. By (7.1) Z' determines a partition $\{S, Y-S\}$ of Y such that $S\subset U$. We have $a\in U-S$, by the definition of U. Choose $b\in S$.

By (6.2) $Y\cup Z\cup Z'$ is a plane P of $\boldsymbol{M}$. The line $\langle P-\{b\}\rangle$ of $\boldsymbol{M}$ is on the points $Z\cup(Y-U)$ and $Z'\cup(Y-S)$ of $\boldsymbol{M}$. Since these are distinct and not disjoint their mod 2 sum, Q say, is also a point of $\boldsymbol{M}$ on $\langle P-\{b\}\rangle$. But the mod 2 sum of Z and Z' is the third point of $\boldsymbol{M}\cdot(M-Y)$ on the line $Z\cup Z'$. It determines the partition $\{Q\cap Y, Y-(Q\cap Y)\}=\{U-S, Y-(U-S)\}$ of Y. This contradicts the definition of U.

We deduce from this contradiction that U is itself a member of $\pi(\boldsymbol{M}, B, Y)$ and that it meets no other member. Since a may be any cell of Y the theorem follows.

While discussing the next three theorems we bear in mind that the minors of $\boldsymbol{M}$ are regular matroids (3.7).

(7.4) *Let B be a bridge of Y in $\boldsymbol{M}$ and S a subset of M such that $B\cup Y\subseteq S$. Then B is a bridge of Y in $\boldsymbol{M}\times S$, and $\pi(\boldsymbol{M}\times S, B, Y)=\pi(\boldsymbol{M}, B, Y)$.*

Proof. $(\boldsymbol{M}\times S)\cdot(S-Y)=(\boldsymbol{M}\cdot(M-Y))\times(S-Y)$, by (3.4). Hence B is a separator of $(\boldsymbol{M}\times S)\cdot(S-Y)$, by (3.8). Moreover $((\boldsymbol{M}\times S)\cdot(S-Y))\times B =((\boldsymbol{M}\cdot(M-Y))\times(S-Y))\times B=(\boldsymbol{M}\cdot(M-Y))\times B$, by (3.1), and this matroid is connected. Hence B is a bridge of Y in $\boldsymbol{M}\times S$. Since $(\boldsymbol{M}\times S)\times(B\cup Y) =\boldsymbol{M}\times(B\cup Y)$, by (3.1), the theorem follows.

(7.5) *Let B be a bridge of Y in $\boldsymbol{M}$. Let S be a subset of M such that $B\cup Y\subseteq S$ and no $Z\in\boldsymbol{M}\cdot(M-Y)$ is a subset of $M-S$. Then $Y\in\boldsymbol{M}\cdot S$, B is a bridge of Y in $\boldsymbol{M}\cdot S$, and $\pi(\boldsymbol{M}\cdot S, B, Y)=\pi(\boldsymbol{M}, B, Y)$.*

Proof. $(\boldsymbol{M}\cdot S)\cdot(S-Y)=(\boldsymbol{M}\cdot(M-Y))\cdot(S-Y)$ by (3.2). Hence B is a separator of $(\boldsymbol{M}\cdot S)\cdot(S-Y)$, by (3.8). Moreover $((\boldsymbol{M}\cdot S)\cdot(S-Y))\cdot B$ $=(\boldsymbol{M}\cdot(M-Y))\cdot B$, by (3.2), and this matroid is connected. Hence B is a bridge of Y in $\boldsymbol{M}\cdot S$ if $Y\in\boldsymbol{M}\cdot S$.

Suppose some $X\in\boldsymbol{M}\times((M-S)\cup B\cup Y)$ meets $M-S$. Then, by HI, (3.1), some $Z\in\boldsymbol{M}\cdot(M-Y)$ meets $M-S$ and satisfies $Z\subseteq(M-S)\cup B$. But then $Z\subseteq M-S$, since B is a separator of $\boldsymbol{M}\cdot(M-Y)$, and this is contrary to the definition of S. We thus have $(\boldsymbol{M}\cdot S)\times(B\cup Y)=(\boldsymbol{M}\times((M-S)\cup B\cup Y))\cdot(B\cup Y)$, by (3.3), $=(\boldsymbol{M}\times(B\cup Y))\cdot(B\cup Y)=\boldsymbol{M}\times(B\cup Y)$. We note that this implies $Y\in M\cdot S$. The theorem follows.

(7.6) *Let B be any bridge of Y in $\boldsymbol{M}$. Let S be a subset of M such that $M-Y\subset S$. Then $S\cap Y\in\boldsymbol{M}\cdot S$, B is a bridge of $S\cap Y$ in $\boldsymbol{M}\cdot S$, and*

$$\pi(\boldsymbol{M}\cdot S,\ B,\ S\cap Y)$$

is the class of all non-null intersections with S of members of $\pi(\boldsymbol{M}, B, Y)$.

Proof. There exists $Y'\in\boldsymbol{M}\cdot S$ such that $Y'\subseteq S\cap Y$ and $Y''\in\boldsymbol{M}$ such that $Y''\cap S=Y'$. But then $Y''\subseteq Y$. Hence $Y''=Y$ and $S\cap Y=Y'\in\boldsymbol{M}\cdot S$ by Axiom I.

$(\boldsymbol{M}\cdot S)\cdot(S-(S\cap Y))=\boldsymbol{M}\cdot(S-(S\cap Y))=\boldsymbol{M}\cdot(M-Y)$, by (3.2). Hence B is a bridge of $S\cap Y$ in $M\cdot S$.

Suppose $W\in\pi(M, B, Y)$ and $W\cap S\neq\emptyset$. By (7.3) there exists $X\in\boldsymbol{M}\times(B\cup Y)$ such that $X\cap Y=W$. By HII, (3.1), there exists $X'\in(\boldsymbol{M}\cdot S)\times(B\cup(S\cap Y))$ such that $\emptyset\subset X'\cap Y\subseteq X\cap S\cap Y=W\cap S$. Hence there exists $W'\in\pi(\boldsymbol{M}\cdot S, B, S\cap Y)$ such that $W'\subseteq W\cap S$.

Conversely, suppose $W'\in\pi(\boldsymbol{M}\cdot S, B, S\cap Y)$. By (7.3) there exists $X'\in(\boldsymbol{M}\cdot S)\times(B\cup(S\cap Y))$ such that $X'\cap(S\cap Y)=W'$. Hence there exists $X\in\boldsymbol{M}\times(B\cup Y)$ such that $X\cap(S\cap Y)=W'$. Hence, by (7.3), there exists $W\in\pi(\boldsymbol{M}, B, Y)$ such that $\emptyset\subset W\cap S\subseteq W'$.

The foregoing results, together with (7.3), imply (7.6).

(7.7) *Let B_1 and B_2 be distinct bridges of Y in $\boldsymbol{M}$. Let $W_1\in\pi(\boldsymbol{M}, B_1, Y)$ and $W_2\in\pi(\boldsymbol{M}, B_2, Y)$ be such that $W_2\subseteq W_1$. Let $Z\in\boldsymbol{M}\cdot(M-Y)$ on B_2 be such that $Z\cup W_2\in\boldsymbol{M}$, so that $Z\cup(Y-W_2)$, $=Y'$ say, is a point of $\boldsymbol{M}$. Then B_1 is a bridge of Y' in $\boldsymbol{M}$.*

Proof. $(\boldsymbol{M}\cdot(M-Y'))\cdot B_1=(\boldsymbol{M}\cdot(M-Y))\cdot B_1$, by (3.2). As this matroid is connected B_1 is a subset of a bridge B' of Y' in $\boldsymbol{M}$, by (3.8).

Suppose $Z_1\in\boldsymbol{M}\cdot(M-Y')$ on B' and $Z_1\cap B_1\neq\emptyset$. There exists $X\in\boldsymbol{M}\times(B'\cup Y')$ such that $X\cap B'=Z_1$. By HII, (3.1), there exists $Z_2\in\boldsymbol{M}\cdot(M-Y)$ such that $Z_2\cap B_1\neq\emptyset$ and $Z_2\subseteq X\cap(M-Y)$. This implies $Z_2\cap B_1\subseteq Z_1$. But B_1 is a separator of $\boldsymbol{M}\cdot(M-Y)$, and so $Z_2\subseteq Z_1\cap B_1$. Since Z_2 is on B_1 it determines a partition $\{S, T\}$ of Y such that $W_2\subseteq W_1\subseteq T$. Since $Z_2\cup S\in\boldsymbol{M}$ there exists $Z_3\in\boldsymbol{M}\cdot(M-Y')$ such that $Z_3\subseteq(Z_2\cup S)\cap(M-Y')\subseteq Z_2\subseteq Z_1\cap B_1$. Then $Z_1=Z_3$, by Axiom I, and so $Z_1\subseteq B_1$. We deduce that B_1 is a non-null separator of $(\boldsymbol{M}\cdot(M-Y'))\times B'$, and therefore $B_1=B'$.

We write $\beta(\boldsymbol{M}, Y)$ for the number of bridges of Y in $\boldsymbol{M}$. An *n-bridge* of Y in $\boldsymbol{M}$ is a bridge B of Y in $\boldsymbol{M}$ such that $\alpha(\boldsymbol{\pi}(\boldsymbol{M}, B, Y))=n$. Two n-bridges B_1 and B_2 of Y in M are *equivalent* if $\boldsymbol{\pi}(\boldsymbol{M}, B_1, Y)=\pi(\boldsymbol{M}, B_2, Y)$. Two bridges B_1 and B_2 of Y in $\boldsymbol{M}$ *overlap* if the equation $W_1\cup W_2=Y$ is false whenever $W_1\in\boldsymbol{\pi}(\boldsymbol{M}, B_1, Y)$ and $W_2\in\boldsymbol{\pi}(\boldsymbol{M}, B_2, Y)$. For example any two skew bridges overlap.

(7.8) *If $\boldsymbol{M}$ is any regular matroid such that $d\boldsymbol{M}\geqq 2$ then there exists $Y\in\boldsymbol{M}$ such that $\beta(\boldsymbol{M}, Y)\geqq 2$.*

Proof. Assume the theorem false. Choose $Y\in\boldsymbol{M}$. Then $\beta(\boldsymbol{M}, Y)=1$ since $d\boldsymbol{M}>0$. Thus $M-Y$ is a bridge of Y in $\boldsymbol{M}$.

Choose $W\in\boldsymbol{\pi}(\boldsymbol{M}, M-Y, Y)$. There exists $Z\in\boldsymbol{M}\cdot(M-Y)$ determining the partition $\{W, Y-W\}$ of Y, by (6.1) and (7.3). Then $Y\cup Z$ is a line of $\boldsymbol{M}$, possibly disconnected, and $Z\cup(Y-W)$, $=Y'$ say, is a point of $\boldsymbol{M}$. We have $M-(Y\cup Z)\neq\emptyset$ since $d\boldsymbol{M}>1$.

By (6.2) W is a point of $\boldsymbol{M}\cdot(M-Y')$. Suppose $Z'\in\boldsymbol{M}\cdot(M-Y')$ and $Z'\cap W\neq\emptyset$. Then there exists $X\in\boldsymbol{M}$ such that $Z'=X\cap(M-Y')$ and so $X\cap W=Z'\cap W$. Hence $W\subseteq Z'$ since $W\in\pi(\boldsymbol{M}, M-Y, Y)$, and so $Z'=W$ by Axiom I. Hence W is a separator of $\boldsymbol{M}\cdot(M-Y')$. But $(M-Y')-W$ $=M-(Y\cup Z)\neq\emptyset$. Hence $\beta(\boldsymbol{M}, Y')\geqq 2$, contrary to assumption.

(7.9) *Let B_1 and B_2 be overlapping bridges of a point Y of the regular matroid $\boldsymbol{M}$. Then either B_1 and B_2 are skew with respect to Y or they are equivalent 3-bridges.*

Proof. Suppose B_1 and B_2 are not equivalent n-bridges. Without loss of generality we may suppose $\emptyset\subset W_1\cap W_2\subset W_2$ for some $W_1\in\boldsymbol{\pi}(\boldsymbol{M}, B_1, Y)$ and $W_2\in\boldsymbol{\pi}(\boldsymbol{M}, B_2, Y)$. We can find $U\in\boldsymbol{\pi}(\boldsymbol{M}, B_1, Y)$ such that $\emptyset\subset U$ $\cap(Y-W_2)\subset Y-W_2$. For otherwise some $U'\in\boldsymbol{\pi}(\boldsymbol{M}, B_1, Y)$ satisfies $Y-W_2$ $\subseteq U'$, whence $U'\cup W_2=Y$, contrary to hypothesis. Thus, by (7.3), there are points Z_1 and Z_2 of $\boldsymbol{M}\cdot(M-Y)$ on B_1 cutting W_2 and $Y-W_2$ respectively. By (5.1) and (6.4) some point of $\boldsymbol{M}\cdot(M-Y)$ on B_1 cuts both these sets. Hence B_1 and B_2 are skew, by (7.3).

Suppose next that B_1 and B_2 are equivalent n-bridges. Then $n\geqq 3$ since the bridges overlap. If $n>3$ we choose distinct members W_1, W_2, W_3 and W_4 of $\boldsymbol{\pi}(\boldsymbol{M}, B_1, Y)=\boldsymbol{\pi}(\boldsymbol{M}, B_2, Y)$. Applying (5.1), (6.4) and (7.3) as before we find there exists $Z\in\boldsymbol{M}\cdot(M-Y)$ on B_1 cutting both $W_1\cup W_2$ and $W_3\cup W_4$. We may suppose without loss of generality that Z determines a partition $\{S, T\}$ of Y such that $W_1\cup W_3\subseteq S$ and $W_2\cup W_4\subseteq T$. A similar argument shows that some $Z'\in\boldsymbol{M}\cdot(M-Y)$ on B_2 cuts both $W_1\cup W_3$ and $W_2\cup W_4$. Hence B_1 and B_2 are skew.

8. **Proof of the main theorem.** Let G be any graph and Y any point of $\boldsymbol{B}(G)$. Of the components of $G:(E(G)-Y)$ just two include ends in G of members of Y. We call these two the *end-graphs* of Y in G.

Let B be any bridge of Y in $\boldsymbol{B}(G)$. It is an elementary separator of $\boldsymbol{B}(G)\cdot(E(G)-Y)$, that is $\boldsymbol{B}(G\cdot(E(G)-Y))$ by (4.8). Hence $G\cdot B$ is a separate of $G\cdot(E(G)-Y)$, by (4.11). It follows that $G\cdot B$ is a separate of some component H of $G:(E(G)-Y)$.

If $v\in V(G\cdot B)$ we write $C(B, v)$ for the component of $H:(E(H)-B)$ having v as a vertex. We also write $Y(B, v)$ for the set of all $A\in Y$ such that one end of A in G is a vertex of $C(B, v)$.

We recall that $B(G)$ is regular, by (4.1).

(8.1) *If $G\cdot B$ is a separate of an end-graph H of Y in G then $\pi(\boldsymbol{B}(G), B, Y)$ is the class of all $Y(B, v)$ such that $v\in V(G\cdot B)$ and $Y(B, v)\neq\emptyset$. In the remaining case $\pi(\boldsymbol{B}(G), B, Y)=\{Y\}$.*

Proof. Let H be the component of $G:(E(G)-Y)$ having $G\cdot B$ as a separate. Suppose $X\in\boldsymbol{B}(G)\times(B\cup Y)$. We can write $X=Q(T, U)$ for suitable complementary subsets T and U of $V(G)$. If K is a component other than H of $G:(E(G)-Y)$ then $v(K)$ is a subset of T or U since $E(K)\cap X=\emptyset$. Similarly $V(C(B, v))$ is a subset of T or U for each $v\in V(G\cdot B)$. If H is not an end-graph of Y it follows that $X\cap Y=\emptyset$ or Y. We then deduce that $\pi(\boldsymbol{B}(G), B, Y)=\{Y\}$. If H is an end-graph of Y we may suppose the vertices of the other end-graph to belong to U. Then $X\cap Y$ is the union of the sets $Y(B, v)$ such that $v\in T\cap V(G\cdot B)$. Moreover if $v\in V(G\cdot B)$ and $Y(B, v)\neq\emptyset$ we can find $A\in Y(B, v)$ and $X\in B(\mathbf{G})\times(B\cup Y)$ such that

$$A \in X \subseteq Q(V(C(B, v)), V(G) - V(C(B, v))),$$

by (4.5). Then $A\in X\cap Y\subseteq Y(B, v)$. It follows that $\pi(\boldsymbol{B}(G), B, Y)$ is the class of all non-null $Y(B, v)$, as required.

Let $\boldsymbol{M}$ be any regular matroid. We say that a point Y of $\boldsymbol{M}$ is *even* if the bridges of Y in $\boldsymbol{M}$ can be arranged in two disjoint classes so that no two members of the same class overlap. If this arrangement is impossible Y is *odd.* Clearly Y is odd if and only if there is a cyclic sequence $P=(B_1, B_2, \cdots, B_{2n+1}, B_1)$ of an odd number $2n+1$, where $n\geqq 1$, of distinct bridges of Y in $\boldsymbol{M}$ such that each B_i overlaps its two neighbours in P. We refer to such a sequence as an *odd overlap-circuit* of Y.

(8.2) *In a graphic matroid every point is even.*

Proof. Let Y be any point of a graphic matroid $\boldsymbol{M}$. There is a graph G such that $\boldsymbol{M}=\boldsymbol{B}(G)$. Let the end-graphs of Y in G be H and K. We arrange the bridges of Y in $\boldsymbol{M}$ in two disjoint classes $\boldsymbol{T}$ and $\boldsymbol{U}$ so that those corresponding to separates of $H(K)$ are in $\boldsymbol{T}(\boldsymbol{U})$. If Y is odd we may suppose that two distinct members B_1 and B_2 of $\boldsymbol{T}$ overlap. Then $G\cdot B_1$ and $G\cdot B_2$ are separates of H. There are vertices v_1 of $G\cdot B_1$ and v_2 of $G\cdot B_2$ such that $G\cdot B_2$ is a subgraph of $C(B_1, v_1)$ and $G\cdot B_1$ is a subgraph of $C(B_2, v_2)$. It is clear that each vertex of H is a vertex of $C(B_1, v_1)$ or $C(B_2, v_2)$. Hence $Y(B_1, v_1)\cup Y(B_2, v_2) = Y$, contrary to the supposition that B_1 and B_2 overlap.

(8.3) *The circuit-matroids of the Kuratowski graphs are not graphic.*

Proof. For the complete 5-graph, as defined in §4, we find by inspection that $(\{L_{13}\}, \{L_{24}\}, \{L_{35}\}, \{L_{14}\}, \{L_{25}\}, \{L_{13}\})$ is an odd overlap-circuit of the point $\{L_{12}, L_{23}, L_{34}, L_{45}, L_{15}\}$ of the circuit-matroid. For the Thomsen graph $(\{L_{12}\}, \{L_{23}\}, \{L_{31}\}, \{L_{12}\})$ is an odd overlap-circuit of the point $\{L_{11}, L_{21}, L_{22}, L_{32}, L_{33}, L_{13}\}$ of the circuit-matroid. (See Figure I.) Hence in each case the circuit-matroid is nongraphic, by (8.2).

(8.4) *Let Y be a point of a connected graphic matroid $\boldsymbol{M}$ such that no two bridges of Y in $\boldsymbol{M}$ overlap. Then there exists a nonseparable graph G such that $\boldsymbol{M}=\boldsymbol{B}(G)$ and $Y=Q(\{a\}, V(G)-\{a\})$ for some $a\in V(G)$.*

Proof. There exists G such that $\boldsymbol{M}=\boldsymbol{B}(G)$. We take G to be nonseparable. This is possible by (4.11) since $\boldsymbol{M}=\boldsymbol{B}(G)\cdot E(G)=\boldsymbol{B}(G\cdot E(G))$ by (4.8). Let the end-graphs of Y in G be G_1 and G_2. We suppose G, G_1 and G_2 chosen so that $\alpha_1(G_2)$ has the least possible value.

Assume $\alpha_1(G_2)>0$. Then $\alpha_1(G_1)>0$. Choose separates $G\cdot B_1$ and $G\cdot B_2$ of G_1 and G_2 respectively. B_1 and B_2 are bridges of Y in $\boldsymbol{M}$, by (4.8) and (4.11). Since they do not overlap there are, by (8.1), vertices v_1 and v_2 of $G\cdot B_1$ and $G\cdot B_2$ respectively such that

$$Y(B_1, v_1) \cup Y(B_2, v_2) = Y. \tag{i}$$

Keeping B_2 and v_2 fixed we consider all possible choices of B_1 and v_1 for which (i) is true, and we select one for which $C(B_1, v_1)$ has the least possible number of edges.

Let $P_1, P_2, \cdots, P_k$, where $P_1=B_1$, be the sets of edges of the separates of G_1 having v_1 as a vertex. (See Figure II.) For each P_j let E_j be the subgraph of G_1 which is the union of $G\cdot P_j$ and those subgraphs $C(P_j, x)$ of G_1 such that $x\in V(G\cdot P_j)-\{v_1\}$. The graphs E_j have the common vertex v_1. But by the definition of a separate no two of them have any other common vertex. Since G_1 is connected it is the union of the graphs E_j.

By hypothesis we can, for each P_j, find a vertex p_j of $G\cdot P_j$ and a vertex q_j of $G\cdot B_2$ such that

$$Y(P_j, p_j) \cup Y(B_2, q_j) = Y. \tag{ii}$$

In accordance with (i) we take $p_1=v_1$ and $q_1=v_2$.

Since G is nonseparable it follows from (ii) that for each j there is an edge a_j in $Y(B_2, q_j)$ but not in $Y(P_j, p_j)$ and an edge b_j in $Y(P_j, p_j)$ but not in $Y(B_2, q_j)$.

Suppose $p_j\neq v_1$. Considering the edge a_1, of which one end is a vertex of E_1 other than v_1, we find that $q_j=q_1=v_2$, by (ii). But in this case $C(P_j, p_j)$ is a subgraph of $C(B_1, v_1)$ having fewer edges than $C(B_1, v_1)$. This contradicts the definition of B_1 and v_1. We deduce that $p_j=v_1$ for each j.

Considering the edge a_j we see that q_j is uniquely determined for each P_j. Let Z_j denote the set of all members of Y having one end a vertex of E_j other than v_1. Then Z_j is non-null since it includes a_j. By (ii) each member of Z_j has its other end a vertex of $C(B_2, q_j)$.

If $x\in V(G\cdot B_2)$ we denote by $R(x)$ the subgraph of G formed by taking the union of $C(B_2, x)$ and those graphs E_j for which $q_j=x$, and then adjoining the members of the corresponding sets Z_j as new edges.

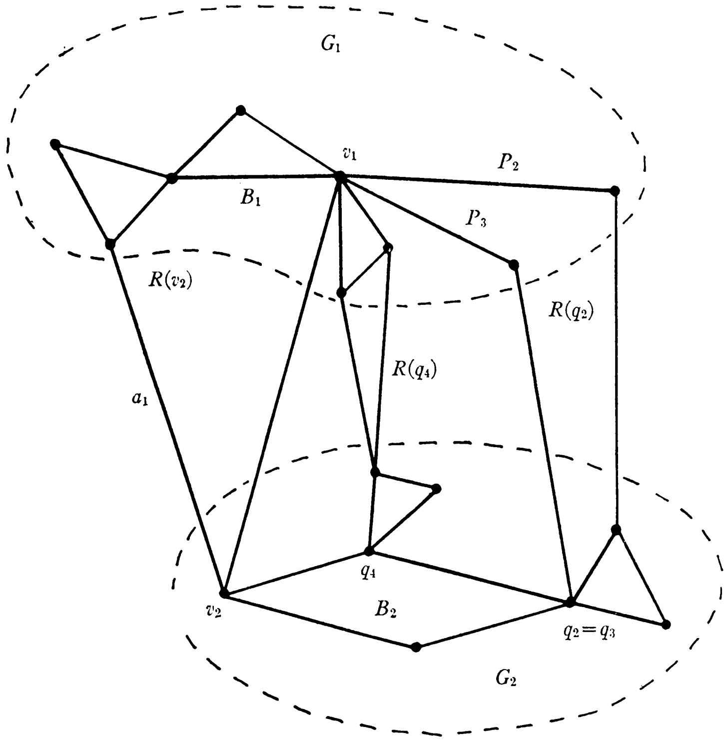

FIG. II

For a given x the graph $R(x)$ may have only one vertex. If not the set $E(R(x))$ is non-null and its vertices of attachment in G are x and v_1. For since G is nonseparable there are at least two such vertices of attachment, and x and v_1 are the only possibilities. Moreover if x and y are distinct vertices of $G\cdot B_2$ then $R(x)$ and $R(y)$ have at most one common vertex (v_1) and no common edge. We may therefore reverse all the non-null sets

$$E(R(x)),\ x \in V(G\cdot B_2),$$

without mutual inteference. By (4.12) G is then transformed into another graph G' with the same bond-matroid $\boldsymbol{M}$. Since G' has no isolated vertices we have $G'\cdot E(G') = G'$. Hence G' is nonseparable, by (4.11). But the transformation replaces G_1 by an end-graph H of Y in G' such that $E(H) = E(G_2) - B_2 \subset E(G_2)$. This is contrary to the choice of G, G_1 and G_2.

We deduce that $\alpha_1(G_2) = 0$. The theorem follows.

For any binary matroid $\boldsymbol{M}$ on a set M there is a binary chain-group N on M such that $\boldsymbol{M} = \boldsymbol{M}(N)$. Any mod 2 sum K of points of $\boldsymbol{M}$ must be the domain of a chain of N. Hence if K is non-null it must have some point of $\boldsymbol{M}$ as a subset. We make use of this observation in the proof of the next two theorems.

(8.5) *Let* $\boldsymbol{M}$ *be a connected regular matroid on a set* M. *Let* Y *be an even point of* $\boldsymbol{M}$ *such that* $\beta(\boldsymbol{M}, Y) \geqq 1$ *and* $\boldsymbol{M}\times(B\cup Y)$ *is graphic for each bridge* B *of* Y *in* $\boldsymbol{M}$. *Then* $\boldsymbol{M}$ *is graphic.*

Proof. If possible choose $\boldsymbol{M}$ and Y so that the theorem fails and $\alpha(M)$ has the least value consistent with this condition. Then $\beta(\boldsymbol{M}, Y) \geqq 2$, since otherwise $\boldsymbol{M}=\boldsymbol{M}\times(B\cup Y)$, where B is the only bridge of Y in $\boldsymbol{M}$, and the theorem is trivially true.

Since Y is even we can classify the bridges of Y in $\boldsymbol{M}$ in two disjoint classes $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ so that no two members of the same class overlap. Since $\beta(\boldsymbol{M}, Y) \geqq 2$ we can arrange that $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ are both non-null. These conditions being satisfied we write U_1 and U_2 for the unions of the members of $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ respectively.

Let B be any bridge of Y in $\boldsymbol{M}$. As B is not a separator of $\boldsymbol{M}$ it meets some point of $\boldsymbol{M}\cdot(M-Y)$. It must therefore be a connected flat of $\boldsymbol{M}\cdot(M-Y)$. The matroid $\boldsymbol{M}\times(B\cup Y)$ is connected. For otherwise we have $\boldsymbol{M}\cdot B = (\boldsymbol{M}\cdot(M-Y))\cdot B = (\boldsymbol{M}\cdot(M-Y))\times B = \boldsymbol{M}\times B$, by (3.2), (3.9) and (6.3), contrary to the hypothesis that $\boldsymbol{M}$ is connected. From this result it follows that $\boldsymbol{M}\times(U_i\cup Y)$ is connected ($i=1, 2$). But $\boldsymbol{M}\times(U_i\cup Y)$ is regular, by (3.7). Further the bridges of Y in $\boldsymbol{M}\times(U_i\cup Y)$ are the members of $\boldsymbol{U}_i$ and no two of them overlap, by (7.4). Moreover for each $B\in\boldsymbol{U}_i$ the matroid $(\boldsymbol{M}\times(U_i\cup Y))\times(B\cup Y)$ is identical with $\boldsymbol{M}\times(B\cup Y)$ and is therefore graphic. Hence $\boldsymbol{M}\times(U_i\cup Y)$ is graphic, by the choice of $\boldsymbol{M}$.

By (8.4) there exists a nonseparable graph G_i and a vertex w_i of G_i such that $\boldsymbol{B}(G_i)=\boldsymbol{M}\times(U_i\cup Y)$ and such that $Y=Q(\{w_i\}, V(G_i)-\{w_i\})$ in G_i.

We may take $V(G_1)$ and $V(G_2)$ to be disjoint subsets of some larger set V. We construct a graph G' as follows. $E(G')=M$ and $V(G')=(V(G_1)\cup V(G_2)) - \{w_1, w_2\}$. If $A\in M-Y$ the ends of A in G' are its ends in G_1 or G_2, but if $A\in Y$ its ends in G' are its end other than w_1 in G_1 and its end other than w_2 in G_2. (See Figure III.)

From this construction it follows that $G':(M-Y)$ has just two components H_1 and H_2, H_i being the end-graph of Y in G_i not including w_i. Hence $Y\in\boldsymbol{B}(G')$. We note that U_i is a separator of $\boldsymbol{B}(G')\cdot(M-Y)=\boldsymbol{B}(G'\cdot(M-Y))$ as well as of $\boldsymbol{M}\cdot(M-Y)$.

To change G_i into $G'\times(U_i\cup Y)$ we have only to replace w_i by H_i and each $v\in V(G_i)-\{w_i\}$ by the edgeless subgraph of G' having v as its only vertex, arranging that corresponding vertices in the two graphs are incident with the same edges. We therefore have $\boldsymbol{B}(G')\times(U_i\cup Y)=\boldsymbol{B}(G'\times(U_i\cup Y)) = \boldsymbol{B}(G_i)=\boldsymbol{M}\times(U_i\cup Y)$, by (4.9). This implies that each $\boldsymbol{B}(G')\times(U_i\cup Y)$ is connected and hence that $B(G')$ is connected.

The mod 2 sum of the three points on any line of a binary matroid is null, by HII, (2.6). Hence the points of $\boldsymbol{B}(G')$ expressible as mod 2 sums of points of $\boldsymbol{M}$ constitute a convex subclass $\boldsymbol{C}$ of $\boldsymbol{B}(G')$. If $\boldsymbol{B}(G')-\boldsymbol{C}$ is not null there exists $X\in\boldsymbol{B}(G')-\boldsymbol{C}$ such that $X\cup Y$ is a line of $\boldsymbol{B}(G')$, by HI, (5.1). Then $X-(X\cap Y)$ is a point of $\boldsymbol{B}(G')\cdot(M-Y)$, by (6.2). Since U_1 and U_2 are

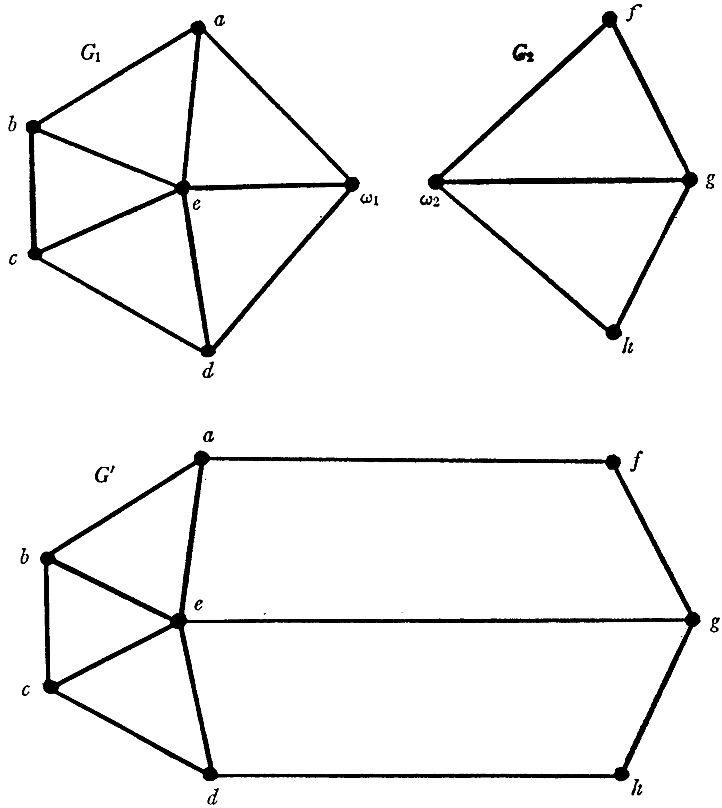

Fig. III

separators of this matroid we have $X \in \boldsymbol{B}(G') \times (U_j \cup Y)$, where $j=1$ or 2. But then $X \in \boldsymbol{M} \times (U_j \cup Y) \subset \boldsymbol{M}$, contrary to the definition of X.

We deduce that each point of $\boldsymbol{B}(G')$ belongs to **C**. Hence for each $X \in \boldsymbol{B}(G')$ there exists $X' \in \boldsymbol{M}$ such that $X' \subseteq X$. A similar argument with $\boldsymbol{M}$ and $\boldsymbol{B}(G')$ interchanged shows that for each $X' \in \boldsymbol{M}$ there exists $X \in \boldsymbol{B}(G')$ such that $X \subseteq X'$. Hence $\boldsymbol{M} = \boldsymbol{B}(G')$ by Axiom I. This is impossible, by the choice of $\boldsymbol{M}$. The theorem follows.

(8.6) *Let* **M** *be a regular matroid on a set M such that* **M** *is not graphic but all the other minors of* **M** *are graphic. Then* **M** *is the circuit-matroid of a Kuratowski graph.*

Proof. Suppose $\boldsymbol{M}$ is not connected. We can find complementary non-null separators S_1 and S_2 of $\boldsymbol{M}$. There exist graphs G_1 and G_2 such that $\boldsymbol{B}(G_1) = \boldsymbol{M} \times S_1$ and $\boldsymbol{B}(G_2) = \boldsymbol{M} \times S_2$. We may take $V(G_1)$ and $V(G_2)$ to be disjoint subsets of some larger set V. Then G_1 and G_2 together constitute a graph G. We note that S_1 and S_2 are separators of $\boldsymbol{B}(G)$.

If $X \in \boldsymbol{B}(G)$ we have $X \in \boldsymbol{B}(G) \times S_i = \boldsymbol{B}(G) \cdot S_i = \boldsymbol{B}(G \cdot S_i) = \boldsymbol{B}(G_i) = \boldsymbol{M} \times S_i$, where $i=1$ or 2. Hence $X \in \boldsymbol{M}$. We find similarly that if $X \in \boldsymbol{M}$ then $X \in \boldsymbol{B}(G)$.

Hence $\boldsymbol{M}=\boldsymbol{B}(G)$, contrary to hypothesis.

We deduce that $\boldsymbol{M}$ is connected.

Suppose $d\boldsymbol{M}=-1$. Then $\alpha(M)=0$ or 1 since $\boldsymbol{M}$ is connected. We thus have $\boldsymbol{M}=\boldsymbol{B}(G)$ for some graph G with just one vertex. If $\alpha(M)=0$, G is edgeless; otherwise $E(G)$ consists of a single loop.

If $d\boldsymbol{M}=0$ we have $M\in\boldsymbol{M}$. Then $\boldsymbol{M}=\boldsymbol{B}(G)$ where G has just two vertices and each $A\in M$ is an edge of G incident with both of them.

Suppose $d\boldsymbol{M}=1$. Then M is a connected line of $\boldsymbol{M}$. Let its three points be X_1, X_2 and X_3. Each of these is the mod 2 sum of the others. Hence $X_1\cap X_2\cap X_3=\varnothing$. But $X_1\cap X_2$, $X_2\cap X_3$ and $X_3\cap X_1$ are all non-null, by Axiom I, and their union is M. Let G be a graph with just three vertices v_1, v_2 and v_3, and such that $E(G)=M$. We take each member of $X_i\cap X_j$ to be incident with v_i and v_j $(1\leqq i<j\leqq 3)$. It is readily verified that $\boldsymbol{M}=\boldsymbol{B}(G)$.

From the preceding analysis we conlude that $d\boldsymbol{M}\geqq 2$.

LEMMA I. *Suppose $Y\in\boldsymbol{M}$ and $\beta(\boldsymbol{M}, Y)\geqq 2$. Then Y is odd. Moreover the bridges of Y in M can be arranged in a single odd overlap-circuit $F(\boldsymbol{M}, Y)$ so that two bridges overlap only if they are consecutive in $F(\boldsymbol{M}, Y)$.*

Proof. If Y is even $\boldsymbol{M}$ is graphic by (8.5), contrary to hypothesis. Let $F(\boldsymbol{M}, Y)$ be an odd overlap-circuit of Y with as few terms as possible. Let the union of its terms be S. By (7.4) Y is an odd point of $\boldsymbol{M}\times S$. Hence $S=M$, by (8.2). Thus $F(\boldsymbol{M}, Y)$ includes all the bridges of Y in $\boldsymbol{M}$. If any two bridges overlap and are not consecutive in $F(\boldsymbol{M}, Y)$ then Y has an odd overlap-circuit with fewer terms than $F(\boldsymbol{M}, Y)$, which is impossible.

LEMMA II. *Let Y be any odd point of $\boldsymbol{M}$. Suppose some $Z\in\boldsymbol{M}\cdot(M-Y)$ on a bridge B of Y in $\boldsymbol{M}$ is skew to both neighbours of B in $F(\boldsymbol{M}, Y)$. Then B is unicellular.*

Proof. Write $S=M-(B-Z)$. Now $Y\cup Z$ is a line of $\boldsymbol{M}\times S$. Hence $Z\in(\boldsymbol{M}\times S)\cdot(S-Y)$, by (6.2). Moreover Z determines the same partition of Y in $\boldsymbol{M}\times S$ as in $\boldsymbol{M}$. Using (7.4) we deduce that Y has an odd overlap-circuit in $\boldsymbol{M}\times S$ obtained from $F(\boldsymbol{M}, Y)$ by substituting Z for B. Hence $S=M$, that is $B=Z$, by (8.2).

Let Z determine the partition $\{U, V\}=\pi(\boldsymbol{M}, Z, Y)$ of Y. Choose $a\in Z$ and write $T=M-(Z-\{a\})$. We have $Y\in\boldsymbol{M}\cdot T$, by (7.5). Now $\{a\}$ is a point of $(\boldsymbol{M}\cdot T)\cdot(T-Y)$ and a bridge of Y in $\boldsymbol{M}\cdot T$. There exists $X'\in\boldsymbol{M}\cdot T$ such that $a\in X'\subseteq Z\cup U\in\boldsymbol{M}$, by HII, (3.1). There exists $X\in\boldsymbol{M}$ such that $X\cap T=X'$. By Axiom I, $X=Z\cup U$ and so $X'=U\cup\{a\}$. Hence $\{a\}$ determines the same partition $\{U, V\}$ of Y in $\boldsymbol{M}\cdot T$ as Z does in $\boldsymbol{M}$. Using (7.5) we deduce that Y has an odd overlap-circuit in $\boldsymbol{M}\cdot T$ obtained from $F(\boldsymbol{M}, Y)$ by substituting $\{a\}$ for Z. Hence $T=M$, that is $B=Z=\{a\}$, by (8.2).

LEMMA III. *If $Y\in\boldsymbol{M}$ and $\beta(\boldsymbol{M}, Y)\geqq 5$, then every bridge of Y in $\boldsymbol{M}$ is unicellular.*

Proof. By Lemma I Y is odd. Let B be any bridge of Y in $\boldsymbol{M}$ and let B_1 and B_2 be its two neighbours in $F(\boldsymbol{M}, Y)$. Now B and B_1 are not equivalent 3-bridges, for otherwise B_1 and B_2 would overlap, contrary to Lemma I Hence B is skew to B_1, and similarly to B_2, by (7.9). But B_1 and B_2 are not skew, by Lemma I. Lemma III now follows from (6.5) and Lemma II.

Choose $J \in \boldsymbol{M}$ so that $\beta(\boldsymbol{M}, J)$ has the greatest possible value and $\alpha(J)$ has the greatest value consistent with this. Since $d\boldsymbol{M} \geqq 2$ such a choice is possible and $\alpha(J) > 1$ by the connection of $\boldsymbol{M}$. Moreover $\beta(\boldsymbol{M}, J) \geqq 2$, by (7.8). Hence J is odd, by Lemma I. We can write

$$F(\boldsymbol{M}, J) = (B_1, B_2, \cdots, B_{2n+1}, B_1),$$

where n is an integer $\geqq 1$. When convenient we write B_i also as B_{i+2n+1}.

Let $\boldsymbol{\pi}$ be the class of all non-null sets of the form

$$\bigcap_{i=1}^{2n+1} (W_i),$$

where $W_i \in \boldsymbol{\pi}(\boldsymbol{M}, B_i, J)$. Clearly the members of $\boldsymbol{\pi}$ are disjoint and their union is J.

Lemma IV. *If $U \in \boldsymbol{\pi}$, then $\alpha(U) = 1$. Moreover there is an integer i satisfying $1 \leqq i \leqq 2n+1$ such that $J - U = W_i \cup W_{i+1}$ for some $W_i \in \boldsymbol{\pi}(\boldsymbol{M}, B_i, J)$ and some $W_{i+1} \in \boldsymbol{\pi}(\boldsymbol{M}, B_{i+1}, J)$.*

Proof. Choose $a \in U$. Now $J - \{a\} \in \boldsymbol{M} \cdot (M - \{a\})$, by (7.6). But $F(\boldsymbol{M}, J)$ is not an odd overlap-circuit of $J - \{a\}$ in $\boldsymbol{M} \cdot (M - \{a\})$, by (8.2). Hence $U = \{a\}$, by (7.6). Further we can find an integer i satisfying $1 \leqq i \leqq 2n+1$ such that B_i and B_{i+1} do not overlap as bridges of $J - \{a\}$ in $\boldsymbol{M} \cdot (M - \{a\})$. The lemma follows.

Lemma V. *If $1 \leqq i \leqq 2n+1$ there is no $W \in \boldsymbol{\pi}(\boldsymbol{M}, B_i, J)$ such that $W \in \boldsymbol{\pi}$.*

Proof. Suppose such a W exists. By (7.3) there exists $Z \in \boldsymbol{M} \cdot (M - J)$ on B_i such that $W \cup Z \in \boldsymbol{M} \times (B_i \cup J)$. If $\alpha(Z) = 1$ we must have $B_i = Z$. Then $\boldsymbol{\pi}(\boldsymbol{M}, B_i, J)$ is the partition $\{W, J - W\}$ of J determined by Z. This is impossible, by Lemma I, since it implies that B_i overlaps none of the other bridges.

If $\alpha(Z) > 1$ we write $J' = (J - W) \cup Z$. Then all the bridges of J in $\boldsymbol{M}$ other than B_i are bridges of J' in $\boldsymbol{M}$, by (7.7), and there is at least one other bridge of J', meeting W, in $\boldsymbol{M}$. But this implies $\beta(\boldsymbol{M}, J') \geqq \beta(\boldsymbol{M}, J)$ and $\alpha(J') > \alpha(J)$, contrary to the definition of J. The lemma follows.

Suppose $n \geqq 2$. Then each bridge of J in $\boldsymbol{M}$ is unicellular, by Lemma III. Hence, for each i, B_i is a point of $\boldsymbol{M} \cdot (M - J)$ determining a partition $\{S_i, T_i\}$ of J. Moreover S_i and T_i are both non-null, since consecutive members of

$F(\boldsymbol{M}, J)$ overlap. Clearly $\pi(\boldsymbol{M}, B_i, J)=\{S_i, T_i\}$. Consecutive members of $F(\boldsymbol{M}, J)$ are skew, by (7.9). We write $S_i=S_{i+2n+1}$, $T_i=T_{i+2n+1}$.

Consider a particular bridge B_i. By Lemma I we can adjust the notation so that $S_j\subset S_i$ or $S_j\subset T_i$ whenever B_j is distinct from and not consecutive with B_i in $F(\boldsymbol{M}, J)$. We choose one B_j of this kind and arrange, by interchanging S_i and T_i if necessary, that $S_j\subset T_i$. Let B_k be the first member of the sequence $(B_j, \cdots, B_{i+2n-1})$ such that $S_{k+1}\cap S_i\neq\varnothing$. Then S_{k+1} meets both S_i and T_i, since B_{k+1} is skew to B_k. Hence B_{k+1} is consecutive with B_i in $F(\boldsymbol{M}, J)$, that is $k=i+2n-1$. Similarly if B_l is the last member of the sequence $(B_{i+2}, \cdots, B_j)$ such that $S_{l-1}\cap S_i\neq\varnothing$ we find that $l=i+2$. It follows that $S_h\subset T_i$ for each B_h not consecutive with B_i in $F(\boldsymbol{M}, J)$, $(h\neq i)$.

By the result just proved we can adjust the notation so that S_i contains neither S_j nor T_j as a subset whenever B_i and B_j are distinct. Then, by Lemma I, $S_i\cap S_j$ is non-null if and only if B_i and B_j are consecutive in $F(\boldsymbol{M}, J)$.

It follows that the $2n+1$ sets $S_i\cap S_{i+1}$ $(1\leqq i\leqq 2n+1)$ are distinct members of π. We proceed to show that they are the only members of π. Choose any $W\in\pi$. By Lemma IV we can write $W=S_i\cap S_{i+1}$, $S_i\cap T_{i+1}$, $T_i\cap S_{i+1}$ or $T_i\cap T_{i+1}$, for some i satisfying $1\leqq i\leqq 2n+1$. In the first case there is nothing to prove. In the second case we observe that S_{i-1} (or S_{i+2n}) meets S_i and is a subset of T_{i+1}. Hence W is the member $S_{i-1}\cap S_i$ (or $S_{i+2n}\cap S_{i+2n+1}$) of π. We dispose of the third case in a similar way, using S_{i+2} instead of S_{i-1}. In the fourth case we have $S_{i+3}\subseteq T_i\cap T_{i+1}=W$, whence $S_{i+3}=W$ and Lemma V is contradicted.

If $n>2$ we write $J'=(J-S_1)\cup B_1$. Then, by (7.7) $J'\in\boldsymbol{M}$ and $B_3, B_4, \cdots, B_{2n}$ are bridges of J' in $\boldsymbol{M}$. By Lemma I, J' is odd and so $\beta(\boldsymbol{M}, J')\geqq 5$. Hence $\beta(\boldsymbol{M}, J')=\alpha(M)-\alpha(J')$, by Lemma III. But $\alpha(J')=\alpha(J)-\alpha(S_{2n+1}\cap S_1)-\alpha(S_1\cap S_2)+1<\alpha(J)$. Hence $\beta(\boldsymbol{M}, J')>\alpha(M)-\alpha(J)=\beta(\boldsymbol{M}, J)$, contrary to the definition of J.

If $n=2$ we write $S_i\cap S_{i+1}=\{L_{i,i+1}\}$ if $1\leqq i\leqq 4$, and $S_5\cap S_1=\{L_{15}\}$ ($\alpha(S_i\cap S_{i+1})=1$ by Lemma IV). We also write $B_i=\{L_{i,i+2}\}$ if $1\leqq i\leqq 3$, $B_4=\{L_{14}\}$ and $B_5=\{L_{25}\}$. We construct a graph G such that $E(G)=M$. We take G to have just five vertices v_1, v_2, v_3, v_4 and v_5, the ends of L_{ij} in G being v_i and v_j. Then G is a complete 5-graph and the notation is that used in §4. We find that $\boldsymbol{M}=\boldsymbol{C}(G)$. To prove this we observe that J and the five sets $B_i\cup S_i$ are points of both $\boldsymbol{M}$ and $\boldsymbol{C}(G)$. Any point X of $\boldsymbol{M}$ is a linear combination mod 2 of these six. For otherwise some linear combination of points of $\boldsymbol{M}$ would be a non-null proper subset of J and would have a point of the binary matroid $\boldsymbol{M}$ as a subset, contrary to Axiom I. Hence X is a mod 2 sum of points of $\boldsymbol{C}(G)$ and so has a point of the binary matroid $\boldsymbol{C}(G)$ as a subset. Similarly each point of $\boldsymbol{C}(G)$ has a point of $\boldsymbol{M}$ as a subset. Hence $\boldsymbol{M}$ is $\boldsymbol{C}(G)$, the circuit-matroid of a Kuratowski graph, by Axiom I.

In the remaining case $n=1$ and we can write $F(\boldsymbol{M}, J)=(B_1, B_2, B_3)$. By Lemma V, B_1, B_2 and B_3 are not mutually equivalent 3-bridges.

Suppose $\pi(\boldsymbol{M}, B_1, J)=\pi(\boldsymbol{M}, B_2, J)=\{W_1, W_2, W_3\}$. Then B_3 is skew to B_1 and B_2, by (7.9). But B_1 and B_2 are clearly not skew. Hence B_3 is unicellular, by (6.5) and Lemma II. Let it determine the partition $\{S, T\}$ of J. We may suppose without loss of generality that $W_1\cap S$ is non-null. Then $W_1\cap S\in\pi$. It is a proper subset of W_1, by Lemma V. By Lemma IV we may write, without loss of generality, $J-(W_1\cap S)=T\cup W_i$, where $i=1$ or 2. (If $J-(W_1\cap S)=S\cup W_i$ then $J=S\cup W_i$, which is impossible since B_1 and B_3 overlap.) But then $W_3\subseteq T$, $W_3\in\pi$, and Lemma V is contradicted.

By (7.9) we may now suppose that B_1, B_2 and B_3 are mutually skew. Applying (6.5) and Lemma II we find that two of them, say B_1 and B_2 are unicellular. Let them determine partitions $\{S_1, T_1\}$ and $\{S_2, T_2\}$ respectively of J.

Assume $\pi(\boldsymbol{M}, B_3, J)=\{W_1, W_2, \cdots, W_k\}$, where $k\geqq 3$.

Suppose $S_1\cap S_2$ is not a subset of any W_i. Then without loss of generality we may assume it meets both W_1 and W_2. But neither W_1 nor W_2 is a subset of $S_1\cap S_2$, by Lemma V. By Lemma IV and the overlapping of the three bridges we have either

$$J-(S_1\cap S_2\cap W_1)=T_1\cup T_2$$

or

$$J-(S_1\cap S_2\cap W_1)=T_x\cup W_u,$$

where $x=1$ or 2 and $2\leqq u\leqq k$. The first alternative must be ruled out since it implies $S_1\cap S_2\cap W_2=\emptyset$. Adopting the second we see that W_1 and W_u are the only members of $\pi(\boldsymbol{M}, B_3, J)$ meeting $S_1\cap S_2$. Hence $W_u=W_2$. A similar argument, with W_2 replacing W_1, gives

$$J-(S_1\cap S_2\cap W_2)=T_y\cup W_1,$$

where $y=1$ or 2. We have $x\neq y$ since otherwise $S_{3-x}\cup T_x=J$ and B_1 is not skew to B_2. But this implies $W_3\subseteq T_1\cap T_2$,which contradicts Lemma V.

We deduce that $S_1\cap S_2\subseteq W_i$ for some i. Similarly each of the sets $S_1\cap T_2$, $T_1\cap S_2$ and $T_1\cap T_2$ is a subset of some member of $\pi(\boldsymbol{M}, B_3, J)$. Hence, since $k\geqq 3$, we can adjust the notation so that $S_1\cap S_2=W_1$, which is contrary to Lemma V.

The above argument shows that B_1, B_2 and B_3 are all 2-bridges. They are all unicellular, by Lemma II and (7.9). We write $\{S_3, T_3\}$ for the partition of J determined by B_3.

Now S_3 meets either both $S_1\cap S_2$ and $T_1\cap T_2$ or both $S_1\cap T_2$ and $T_1\cap S_2$. For otherwise S_3 would not meet all the sets S_1, S_2, T_1 and T_2, which is impossible since B_1, B_2 and B_3 are mutually skew by (7.9). Without loss of generality we suppose S meets both $S_1\cap S_2$ and $T_1\cap T_2$. We may also assume that $S_1 \cap T_2 \subseteq S_3$, by (7.2). Hence T_3 meets both $S_1 \cap S_2$ and $T_1 \cap T_2$, since B_3 is skew to B_1 and B_2, and $S_2 \cap T_1 \subseteq T_3$, by (7.2).

Using Lemma IV we write $S_1 \cap S_2 \cap S_3 = \{L_{11}\}$, $S_1 \cap T_2 \cap S_3 = \{L_{21}\}$, $T_1 \cap T_2 \cap S_3 = \{L_{22}\}$, $T_1 \cap T_2 \cap T_3 = \{L_{32}\}$, $T_1 \cap S_2 \cap T_3 = \{L_{33}\}$ and $S_1 \cap S_2 \cap T_3 = \{L_{13}\}$. We write also $B_1 = \{L_{23}\}$, $B_2 = \{L_{31}\}$ and $B_3 = \{L_{12}\}$. We construct a graph G such that $E(G) = M$. We take G to have just six vertices a_1, a_2, a_3, b_1, b_2, b_3, the ends of L_{ij} being a_i and b_j. Then G is a Thomsen graph and the notation is that used in §4. By an argument like that already used for the complete 5-graph we find $\boldsymbol{M} = \boldsymbol{C}(G)$. Thus $\boldsymbol{M}$ is the circuit-matroid of a Kuratowski graph, and the proof of (8.6) is complete.

We establish the main theorem as follows. If $\boldsymbol{M}$ is a graphic matroid it is regular, by (4.1), and its minors are all graphic, by (4.10). Hence it has no minor which is the circuit-matroid of a Kuratowski graph, by (8.3). Conversely suppose $\boldsymbol{M}$ is a regular matroid having no such minor. Then either $\boldsymbol{M}$ is graphic or it has a non-graphic minor $\boldsymbol{M}_0$ whose minors other than itself are all graphic, by (3.5). But if the second alternative holds $\boldsymbol{M}_0$ is the circuit-matroid of a Kuratowski graph, by (3.7) and (8.6).

To complete the proof we prove the dual of the foregoing result by applying (2.12), (3.6) and (4.2).

9. **Figures and minors.** The main theorems of HI and HII deal with geometrical figures whereas the main theorem of the present paper is expressed in terms of minors. We now show the relationship between the two concepts.

By a *figure* in a matroid $\boldsymbol{M}$ we mean any collection of flats of $\boldsymbol{M}$ which with S and T includes also $\langle S \cap T \rangle$. The figure *of* $\boldsymbol{M}$ is the collection of all flats of $\boldsymbol{M}$. Two figures, possibly in different matroids, are *equivalent* if there is a 1-1 mapping (equivalence) of one onto the other which preserves dimension and inclusion relations (both ways).

Two cells of a matroid $\boldsymbol{M}$ are *equivalent* if no $X \in \boldsymbol{M}$ contains one but not the other.

Two matroids $\boldsymbol{M}$ and $\boldsymbol{M}'$, on sets M and M' respectively, are *isomorphic* if there is a 1-1 mapping f of M onto M' which maps the points of $\boldsymbol{M}$ onto the points of $\boldsymbol{M}'$.

(9.1) *Let $\boldsymbol{M}$ and $\boldsymbol{M}_0$ be matroids on sets M and M_0 respectively, $\boldsymbol{M}_0$ being connected and having no two cells equivalent. Then $\boldsymbol{M}$ has a figure equivalent to that of $\boldsymbol{M}_0$ if and only if it has a minor isomorphic with $\boldsymbol{M}_0$.*

Proof. Suppose the minor $(\boldsymbol{M} \times S) \cdot T$ of $\boldsymbol{M}$ is isomorphic with $\boldsymbol{M}_0$. We apply the theory of HII, §3, as follows. The minor can be written as $(\boldsymbol{M} \times Z) \cdot T$, where Z is a carrier of $(\boldsymbol{M} \times S) \cdot T$ in $\boldsymbol{M} \times S$. The figure of $(\boldsymbol{M} \times Z) \cdot T$ is equivalent to that of M_0, and the (T, Z)-mapping of $\boldsymbol{M} \times S$ maps it onto an equivalent figure in $\boldsymbol{M} \times Z$, and therefore in $\boldsymbol{M}$.

Conversely suppose $\boldsymbol{M}$ has a figure F equivalent to that of $\boldsymbol{M}_0$ under an equivalence f. We may assume $\boldsymbol{M}_0$ is not a null class since otherwise the theorem is trivially true. Hence there exists $S \in F$ such that $f(S) = M_0$, and

$d = dS \geqq 0$. Enumerate the $(d-1)$-flats of F as $U_1, \cdots, U_k$. For each U_i we can find $a_i \in S$ such that $U_i = \langle S - \{a_i\} \rangle$, by HI, §2. We can write the $(d-1)$-flat $f(U_i)$ of M_0 as $\langle M_0 - \{b_i\} \rangle$ for some $b_i \in M_0$. Let T be the set of the k cells a_i. Since $\boldsymbol{M}_0$ has no two equivalent cells there is a 1-1 mapping g, defined by $g(a_i) = b_i$, of T onto M_0.

If $U \in F$, then $U \cap T$ is a flat of $(\boldsymbol{M} \times S) \cdot T$, by HI, (3.1). Conversely if W is any flat of $(\boldsymbol{M} \times S) \cdot T$ let W' be the union of the points of $\boldsymbol{M} \times S$ common to all $\langle S - \{a_i\} \rangle$ such that $a_i \in T - W$. Then $W' \in F$. It is clear that two distinct flats of F cannot have the same intersection with T. For by the equivalence of F and the figure of M_0 they can all (except S) be expressed as geometrical intersections of $(d-1)$-flats of F. Hence the operation of taking intersections with T defines an equivalence of F onto the figure of $(\boldsymbol{M} \times S) \cdot T$. Accordingly g defines an isomorphism of $(\boldsymbol{M} \times S) \cdot T$ and M_0.

We can use this result to express the main theorem (4.5) of HII in terms of minors. For in HII, §4, constructions are given for matroids whose figures are of Type BI or BII, and these matroids are connected and have no two cells equivalent. On the other hand we can state the main theorem of the present paper as follows: a matroid $\boldsymbol{M}$ is graphic (cographic) if and only if it is regular and has no figure equivalent to that of the circuit-matroid (bond-matroid) of a Kuratowski graph.

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